POINCARÉ-FRIEDRICHS INEQUALITIES OF COMPLEXES OF DISCRETE DISTRIBUTIONAL DIFFERENTIAL FORMS

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Abstract. We derive bounds for the constants in Poincaré-Friedrichs inequalities with respect to mesh-dependent norms for complexes of discrete distributional differential forms. A key tool is a generalized flux reconstruction which is of independent interest. The results apply to piecewise polynomial de Rham sequences on bounded domains with mixed boundary conditions.

1. Introduction

Braess and Schöberl introduced differential complexes of distributional finite element spaces, using them as theoretical background for equilibrated a posteriori error estimation in computational electromagnetism [5]. In their seminal publication, these spaces were studied in the language of vector calculus. They considered only finite element spaces of lowest polynomial order over local element patches.

Their idea was studied in the language of differential forms and integrated into the framework of finite element exterior calculus [19]. In finite element exterior calculus (FEEC) [1, 3]) one considers differential complexes of piecewise polynomial differential forms. A major example is the complex of (lowest-order) Whitney forms with respect to a triangulation $\mathcal{T}$ of a domain,

$$\mathcal{P}_1^- \Lambda^0(\mathcal{T}) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \mathcal{P}_1^- \Lambda^n(\mathcal{T}). \tag{1}$$

which serves as a discretization of the L^2 de Rham complex [16, 1, 3]. Higher order finite element de Rham complexes have been addressed too [15, 2, 23, 8].

Let us recapitulate basic concepts of discrete distributional differential forms. Fix a bounded domain Ω with a triangulation $\mathcal{T}$. Arnold, Falk and Winther [1] discuss finite element de Rham complexes

$$\Lambda^0(\mathcal{T}) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \Lambda^n(\mathcal{T}) \tag{2}$$

constituted by conforming spaces of piecewise polynomial differential forms. Their theory includes (1) and its higher order variants. Fix such a differential complex for demonstrative purposes. Generalizing the ideas of Braess and Schöberl, we consider the distributional finite element de Rham complex

$$\Lambda^0_{-1}(\mathcal{T}^n) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \Lambda^n_{-n-1}(\mathcal{T}^n). \tag{3}$$

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Here, $\Lambda^k_{-k-1}(\mathcal{T}^n) := \Lambda^k_{-1}(\mathcal{T}^n) \oplus \Lambda^{k-1}_{-1}(\mathcal{T}^{n-1}) \oplus \cdots \oplus \Lambda^0_{-1}(\mathcal{T}^{n-k})$ is the direct sum of the spaces $\Lambda^k_{-1}(\mathcal{T}^m) := \bigoplus_{C \in \mathcal{T}^m} \Lambda^k(C)$ of formal sums of smooth differential k-forms associated to m-simplices $C \in \mathcal{T}^m$ of the triangulation. As justification of the term *discrete distributional differential form*, we may interpret these as functionals on the space of test forms $C^\infty\Lambda^{k+n-m}(\overline{\Omega})$, acting by $\phi \mapsto \sum_{C \in \mathcal{T}^m} \int_C \langle \omega_C, \operatorname{tr} \phi \rangle$. The exterior derivative between spaces of discrete distributional differential forms is then well-defined in the sense of distributions. The simplicial chain complex

$$\mathcal{C}_n(\mathcal{T}) \xrightarrow{\partial^n} \dots \xrightarrow{\partial^1} \mathcal{C}_0(\mathcal{T}), \tag{4}$$

well-known from algebraic topology [24], is embedded in (3).

The homology groups of these differential complexes are fully understood. The existence of an isomorphism between the homology spaces of the finite element complex (2), the complex of discrete distributional differential forms (3), and the simplicial chain complex (4) has been established in an earlier publication [19]. The homology theory of the latter complex reflects topological properties of the underlying domain. Those isomorphisms also determine the homology spaces of the conforming finite element complex in the presence of non-trivial boundary conditions.

Not only algebraic but also analytical questions are to be addressed in a finite element context. For example, recall that for any conforming finite element complex (2) of Arnold-Falk-Winther-type, one can prove the existence of a constant $\mu_P > 0$ such that for all $\omega \in \mathsf{d}^k\Lambda^k(\mathcal{T})$ there exists $\rho \in \Lambda^k(\mathcal{T})$ with

$$\|\rho\|_{L^2\Lambda^k(\Omega)} \leq \mu_P \|\omega\|_{L^2\Lambda^{k+1}(\Omega)}, \quad \mathsf{d}^k\rho = \omega. \tag{5}$$

The Poincaré-Friedrichs constant μ_P of the finite element complex (2) bounds the norm of the (generalized) solution operator for the finite element equation $\mathsf{d}\rho = \omega$. Additionally, μ_P appears in stability estimates for mixed finite element methods.

This article establishes analogous Poincaré-Friedrichs inequalities for complexes of discrete distributional differential forms. The agenda requires that we agree on a scalar product over the spaces of discrete distributional differential forms. We use

$$\langle \omega, \eta \rangle_{-h} := \sum_{C \in \mathcal{T}^m} h_C^{n-m} \langle \omega_C, \eta_C \rangle_{L^2\Lambda^k(C)}, \quad \omega, \eta \in \Lambda^k_{-1}(\mathcal{T}^m), \tag{6}$$

where we sum over all m-dimensional simplices of $\mathcal{T}$. Thus, (6) is the piecewise L^2 scalar product with elementwise weights depending on the element diameter. One easily sees that (6) reduces to the usual L^2 scalar product on conforming finite element spaces. This construction yields Hilbert space structures over the spaces $\Lambda^k_{-1}(\mathcal{T}^m)$. We bound the Poincaré-Friedrichs constant in terms of the domain, the quality of the triangulation, and the polynomial degree of the finite element spaces.

To begin with, we recall the well-known algebraic duality between the simplicial chain complex (4) and the complex of Whitney forms (1) (see [12, 7]). Equipping the former complex with the scalar product (6) and the latter complex with the usual L^2 scalar product, we recover an analogous duality result on the level of Hilbert complexes. The Poincaré-Friedrichs constant of the Hilbert complex of simplicial chains is bounded by the Poincaré-Friedrichs constant of the complex of Whitney forms, up to a factor depending on the mesh quality.

This leads to bounds for the Poincaré-Friedrichs constants of the other complexes of discrete distributional differential forms. For example, we bound the Poincaré-Friedrichs constant of (3) by the Poincaré-Friedrichs constant of the lowest-order finite element complex (1), up to terms depending only on the polynomial degree and mesh quality. Moreover, the construction of a preimage under the exterior derivative in (3) reduces, using only local operations, to the construction of a preimage in the simplicial chain complex (4). The computational complexity of the latter, however, is comparable to a first-order differential equation over finite element spaces of lowest polynomial order. We have thus reduced a problem on high-order finite element spaces to an analogous problem on a lowest-order finite element space.

This has implications to conforming finite element spaces from which we have commenced our study in the first place. Reconsider the differential equation $\mathsf{d}\rho = \omega$ between conforming finite element spaces of Arnold-Falk-Winther-type of any polynomial degree. As it turns out, local operations reduce solving this equation (in the sense of least squares) to solving the analogous equation between spaces of simplicial chains. Thus, algorithmically solving $\mathsf{d}\rho = \omega$ requires global computation only as difficult as solving $\mathsf{d}\rho = \omega$ between lowest-order Whitney forms. The other operations are local and their stability depends on the polynomial degree.

The remainder of this article is structured as follows. Section 2 gives a summary of discrete distributional differential forms. Section 3 we recapitulates regularity criteria of simplices and triangulations. Section 4 devises Poincare-Friedrichs inequalities for the horizontal and vertical differential operators. and Section 5 we devises Poincare-Friedrichs-type inequalities for complexes of simplicial chains. Finally Section 6 devises Poincare-Friedrichs-type inequalities for complexes of discrete distributional differential forms.

2. Summary

This section outlines discrete distributional differential forms and their homology theory. The reader is referred to [19] for further background and proofs of the cited results.

2.1. **Simplicial Complexes and Triangulations.** We review simplicial complexes and simplicial chain complexes, and their relation to the topology of domains.

We call $C \subseteq \mathbb{R}^n$ an m-simplex if it is the convex hull of $m+1$ affinely independent points, which we call the *vertices* of C, and write $\dim C := m$ for the dimension of C. We write $F \trianglelefteq C$ if $F \subseteq C$ is a simplex whose vertices are also vertices C. In this article, we assume a fixed orientation on each simplex; for technical convenience, we assume the positive Euclidean orientation to be fixed on every n-simplex, whereas this choice is completely arbitrary for other simplices.

If $\mathcal{T}$ is a set of simplices in $\mathbb{R}^n$, then we write $\mathcal{T}^m$ for the subset of m-dimensional simplices of $\mathcal{T}$. We call $\mathcal{T}$ a *simplicial complex* if for all $C \in \mathcal{T}$ and $F \trianglelefteq C$ we already have $F \in \mathcal{T}$ and for all $C, C' \in \mathcal{T}$ we have either $C \cap C' = \emptyset$ or $C \cap C' \trianglelefteq C$. We call $\mathcal{T}$ *n-dimensional* if $\forall C \in \mathcal{T} : \exists T \in \mathcal{T}^n : C \trianglelefteq T$, in which case all simplices in $\mathcal{T}$ have dimension at most n. We call a simplicial complex $\mathcal{U} \subseteq \mathcal{T}$ a *subcomplex* of $\mathcal{T}$. We let $\mathcal{T}^{[m]}$ denote the smallest subcomplex of $\mathcal{T}$ that contains $\mathcal{T}^m$.

Let $\mathcal{T}$ be any p-dimensional simplicial complex. The *space of simplicial m-chains* $\mathcal{C}_m(\mathcal{T})$ for $0 \leq m \leq p$ is the real vector space generated by $\mathcal{T}^m$. The simplices $\mathcal{T}^m$ are the canonical basis of $\mathcal{C}_m(\mathcal{T})$. We define the simplicial boundary operator

$\partial_m : \mathcal{C}_m(\mathcal{T}) \to \mathcal{C}_{m-1}(\mathcal{T})$ as the linear extension of setting

$$\partial_m C := \sum_{\substack{F \trianglelefteq C \\ F \in \mathcal{T}^{m-1}}} o(F,C) F, \quad C \in \mathcal{T}^m,$$

where $o(F,C) = 1$ if C induces the same orientation on F that we have fixed previously, and $o(F,C) = -1$ otherwise. An important property is $\partial_{m-1}\partial_m = 0$.

The *simplicial chain complex of* $\mathcal{T}$ is the differential complex

$$0 \to \mathcal{C}_p(\mathcal{T}) \xrightarrow{\partial_p} \mathcal{C}_{p-1}(\mathcal{T}) \xrightarrow{\partial_{p-1}} \dots \xrightarrow{\partial_1} \mathcal{C}_0(\mathcal{T}) \to 0.$$

For any simplicial subcomplex $\mathcal{U}$ of $\mathcal{T}$ we define the quotient spaces

$$\mathcal{C}_m(\mathcal{T},\mathcal{U}) := \mathcal{C}_m(\mathcal{T}) / \mathcal{C}_m(\mathcal{U}), \quad 0 \leq m \leq p.$$

The simplices $\mathcal{T}^m \setminus \mathcal{U}^m$ are the canonical basis of $\mathcal{C}_m(\mathcal{T},\mathcal{U})$. The operator ∂_m maps chains equivalent up to $\mathcal{C}_m(\mathcal{U})$ to chains equivalent up to $\mathcal{C}_{m-1}(\mathcal{U})$. This leads to a differential operator $\partial_m : \mathcal{C}_m(\mathcal{T},\mathcal{U}) \to \mathcal{C}_{m-1}(\mathcal{T},\mathcal{U})$, and a differential complex, the *simplicial chain complex of* $\mathcal{T}$ *relative to* $\mathcal{U}$:

$$0 \to \mathcal{C}_p(\mathcal{T},\mathcal{U}) \xrightarrow{\partial_p} \mathcal{C}_{p-1}(\mathcal{T},\mathcal{U}) \xrightarrow{\partial_{p-1}} \dots \xrightarrow{\partial_1} \mathcal{C}_0(\mathcal{T},\mathcal{U}) \to 0.$$

We introduce the quotient spaces

$$\mathcal{H}_m(\mathcal{T},\mathcal{U}) := \frac{\ker\left(\partial_m : \mathcal{C}_m(\mathcal{T},\mathcal{U}) \to \mathcal{C}_{m-1}(\mathcal{T},\mathcal{U})\right)}{\operatorname{ran}\left(\partial_{m+1} : \mathcal{C}_{m+1}(\mathcal{T},\mathcal{U}) \to \mathcal{C}_m(\mathcal{T},\mathcal{U})\right)}.$$

We call $\mathcal{H}_m(\mathcal{T},\mathcal{U})$ the *m-th simplicial homology group of* $\mathcal{T}$ *relative to* $\mathcal{U}$. We call $b_m(\mathcal{T},\mathcal{U}) := \dim \mathcal{H}_m(\mathcal{T},\mathcal{U})$ the *m-th simplicial Betti number of* $\mathcal{T}$ *relative to* $\mathcal{U}$, and we call $b_m(\mathcal{T}) := b_m(\mathcal{T},\emptyset)$ the *m-th absolute simplicial Betti number of* $\mathcal{T}$.

The simplicial Betti numbers are relevant for this article in the following manner. Throughout this article, we assume that $\Omega \subseteq \mathbb{R}^n$ is a *bounded Lipschitz domain* [9]. Then $\partial\Omega$ is a topological manifold of dimension $n-1$ without boundary. Throughout this article, we furthermore assume that we are given two topological submanifolds with boundary Γ_T and Γ_N of $\partial\Omega$ with dimension $n-1$ whose intersection is their common boundary. In particular, Γ_T and Γ_N form an essentially disjoint covering of $\partial\Omega$. We call Γ_T the *tangential boundary part* and call Γ_N the *normal boundary part*.

The *topological Betti numbers* $b_m(\Omega,\Gamma_N)$ are topological invariants of Ω and Γ_N. They are defined as the dimensions of the homology groups of the singular chain complex with real coefficients of Ω relative to Γ_N; see [24, Chapter 4, Section 4] for details. One can show [14, Equation (5.26)] that $b_m(\Omega,\Gamma_N) = b_{n-m}(\Omega,\Gamma_T)$.

Assume that $\mathcal{T}$ is a finite simplicial complex that triangulates Ω, which means that $\overline{\Omega}$ is the union of all simplices in $\mathcal{T}$. We assume additionally that we have subcomplexes $\mathcal{U}$ and $\mathcal{V}$ of the triangulation $\mathcal{T}$ such that $\mathcal{U}$ triangulates Γ_N and such that $\mathcal{V}$ triangulates Γ_T. That is, Γ_N is the union of all simplices in $\mathcal{U}$, and Γ_T is the union of all simplices in $\mathcal{V}$. An important result in algebraic topology [24] gives $b_m(\Omega,\Gamma_N) = b_m(\mathcal{T},\mathcal{U}) = b_{n-m}(\Omega,\Gamma_T) = b_{n-m}(\mathcal{T},\mathcal{V})$.

2.2. **Differential forms.** We provide basic facts and notation regarding differential forms; the reader is referred to the exposition in [18] for further background. We let $C^\infty\Lambda^k(\Omega)$ denote the space of *smooth differential forms* on Ω, and let $C^\infty\Lambda^k(\overline{\Omega}) \subseteq C^\infty\Lambda^k(\Omega)$ be the space of restrictions of differential forms in $C^\infty\Lambda^k(\mathbb{R}^n)$ to Ω. Recall the *exterior product* $\wedge : C^\infty\Lambda^k(\Omega) \times C^\infty\Lambda^l(\Omega) \to C^\infty\Lambda^{k+l}(\Omega)$. We observe $C^\infty\Lambda^k(\overline{\Omega}) \wedge C^\infty\Lambda^l(\overline{\Omega}) \subseteq C^\infty\Lambda^{k+l}(\overline{\Omega})$. For $\omega \in C^\infty\Lambda^k(\Omega)$ and $\eta \in C^\infty\Lambda^l(\Omega)$ we have

$$\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega, \quad \mathsf{d}^{k+l}_\Omega(\omega \wedge \eta) = \mathsf{d}^k_\Omega\omega \wedge \eta + (-1)^k\omega \wedge \mathsf{d}^l_\Omega\eta. \tag{7}$$

The Euclidean Riemannian metric of $\mathbb{R}^n$ induces Riemannian metrics over Ω and over all simplices of $\mathcal{T}$ via pullback. We recall the space $L^2\Lambda^k(\Omega)$ of differential k-forms over Ω with coefficients in $L^2(\Omega)$ and denote the scalar product of $L^2\Lambda^k(\Omega)$ by $\langle\cdot,\cdot\rangle_{L^2\Lambda^k(\Omega)}$. The *Hodge star operator* $\star_\Omega : L^2\Lambda^k(\Omega) \to L^2\Lambda^{n-k}(\Omega)$ is a bounded linear mapping uniquely defined by the identity

$$\langle\omega,\eta\rangle_{L^2\Lambda^k(\Omega)} = \int_\Omega \omega \wedge \star_\Omega\eta, \quad \omega,\eta \in L^2\Lambda^k(\Omega). \tag{8}$$

We also recall the *exterior derivative* $\mathsf{d}^k : C^\infty\Lambda^k(\Omega) \to C^\infty\Lambda^{k+1}(\Omega)$. We know that $\mathsf{d}^k C^\infty\Lambda^k(\overline{M}) \subseteq C^\infty\Lambda^{k+1}(\overline{M})$. The *exterior codifferential* is defined as

$$\delta^k_\Omega : C^\infty\Lambda^k(\Omega) \to C^\infty\Lambda^{k-1}(\Omega), \quad \omega \mapsto (-1)^{n(k+1)+1} \star_\Omega \mathsf{d}^{n-k}_\Omega \star_\Omega \omega.$$

Note that $\delta^k_\Omega\omega = (-1)^k \star^{-1}_\Omega \mathsf{d}^{n-k}_\Omega\star_\Omega$ for all $\omega \in C^\infty\Lambda^k(\Omega)$.

We have introduced differential forms over the Lipschitz domain Ω. Completely analogous definitions provide the calculus of differential forms over any simplices $C \in \mathcal{T}$. We can define spaces $C^\infty\Lambda^k(C)$ and $L^2\Lambda^k(C)$ with the canonical meanings, and we can define the exterior product of differential forms over C and the exterior derivative $\mathsf{d}^k_C : C^\infty\Lambda^k(C) \to C^\infty\Lambda^{k+1}(C)$. The Riemannian metric on $\mathbb{R}^n$ induces Riemannian metrics on the simplices in a natural manner; thus, the Hodge star operator $\star_C : L^2\Lambda^k(C) \to L^2\Lambda^{n-k}(C)$ and the codifferential $\delta^k_C : C^\infty\Lambda^k(C) \to C^\infty\Lambda^{k-1}(C)$ are defined. In particular, analogues of the identities (7) and (8) remain valid. The calculus of differential forms over Ω and over the simplices of $\mathcal{T}$ are connected via trace operators. Let $C \in \mathcal{T}$. We have a well-defined *tangential trace* $\mathrm{tr}^k_C : C^\infty\Lambda^k(\overline{\Omega}) \to C^\infty\Lambda^k(C)$, which is defined as the pullback of along the inclusion of C into Ω.

We introduce spaces of smooth differential k-forms over Ω that satisfy either partial tangential boundary conditions or partial normal boundary conditions:

$$C^\infty_T\Lambda^k(\overline{\Omega}) := \left\{\omega \in C^\infty\Lambda^k(\overline{\Omega}) \mid \forall F \in \mathcal{T}, F \subseteq \Gamma_T : \mathrm{tr}^k_F\,\omega = 0\right\},$$
$$C^\infty_N\Lambda^k(\overline{\Omega}) := \left\{\omega \in C^\infty\Lambda^k(\overline{\Omega}) \mid \forall F \in \mathcal{T}, F \subseteq \Gamma_N : \mathrm{tr}^{n-k}_F \star_\Omega\omega = 0\right\}.$$

2.3. **Hilbert Complexes and L^2 de Rham Complexes.** Hilbert complexes provide a general background for the discussion of L^2 de Rham complexes and finite element de Rham complexes [6]. We focus on finite-dimensional Hilbert complexes in this article, but the general theory of Hilbert complexes is helpful to relate the finite element setting with the analytical background.

A *Hilbert complex* is given by a sequence $(X^k)_{k\geq 0}$ of Hilbert spaces and a sequence $(d^k)_{k\geq 0}$ of closed densely-defined unbounded operators $d^k : \mathrm{dom}(d^k) \subseteq X^k \to X^{k+1}$ such that $\mathrm{ran}\, d^{k-1} \subseteq \ker d^k$ holds. This is usually displayed as a diagram:

$$0 \longrightarrow X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \dots$$

We always assume that the operators d^k have closed range. Hence their adjoints d_i^* have closed range too. Then the k-th *harmonic space* $\mathfrak{H}^k = \ker d^k \cap (\operatorname{ran} d^{k-1})^\perp$ is the orthogonal complement of $\operatorname{ran} d^{k-1}$ in $\ker d^k$. As a consequence, we have the orthogonal *abstract Hodge decomposition*

$$X^k = \operatorname{ran} d^{k-1} \oplus \mathfrak{H}^k \oplus (\ker d^k)^\perp.$$

Write d_k^* for the adjoint of the operator d^k. One can show that the adjoint operators d_k^* have closed range and that $\mathfrak{H}^k = \ker d_{k-1}^* \cap (\operatorname{ran} d_k^*)^\perp = \ker d^k \cap \ker d_{k-1}^*$. It follows that $X^k = \operatorname{ran} d^{k-1} \oplus \mathfrak{H}^k \oplus \operatorname{ran} d_k^*$ is an alternative way to write the Hodge decomposition. In the sequel, we consider several instances of Hilbert complexes in which the harmonic spaces $\mathfrak{H}^k$ encode topological information.

Since the operators d^k are assumed to have closed range, there exists $C_P > 0$ such that for all $\omega \in \operatorname{ran} d^k$ there exists $\rho \in \operatorname{dom}(d^k)$ satisfying $\|\rho\|_{X^k} \leq C_P \|\omega\|_{X^{k+1}}$ and $d^k \rho = \omega$. In particular, we have a linear operator $d_k^\dagger : X^{k+1} \to X^k$ whose operator norm is bounded by C_P and that satisfies $d^k d_k^\dagger d^k \rho = d^k \rho$ for all $\rho \in \operatorname{dom}(d^k)$. We call C_P the Poincaré-Friedrichs constant of the Hilbert complex.

The main example for a Hilbert complex is the L^2 de Rham complex. We consider a variant of the L^2 de Rham complex that incorporates partial boundary conditions, following the exposition by Mitrea, Mitrea, and Gol'dshtein [14]. These Hilbert complexes appear naturally in the study of the Hodge-Laplace equation with mixed boundary conditions. We first introduce the scalar products

$$\langle \omega, \eta \rangle_{H\Lambda^k(\Omega)} := \langle \omega, \eta \rangle_{L^2\Lambda^k(\Omega)} + \langle \mathsf{d}_C^k \omega, \mathsf{d}_C^k \eta \rangle_{L^2\Lambda^{k+1}(\Omega)},$$
$$\langle \omega, \eta \rangle_{H^\star\Lambda^k(\Omega)} := \langle \omega, \eta \rangle_{L^2\Lambda^k(\Omega)} + \langle \delta_C^k \omega, \delta_C^k \eta \rangle_{L^2\Lambda^{k+1}(\Omega)},$$

which we initially define for smooth k-forms only. We let $H_T\Lambda^k(\Omega)$ be the closure of $C_T^\infty \Lambda^k\left(\overline{\Omega}\right)$ by $\langle \cdot, \cdot \rangle_{H\Lambda^k(\Omega)}$ and we let $H_N^\star\Lambda^k(\Omega)$ be the closure of $C_N^\infty \Lambda^k\left(\overline{\Omega}\right)$ by $\langle \cdot, \cdot \rangle_{H^\star\Lambda^k(\Omega)}$. One can show that these are dense subspaces of $L^2\Lambda^k(\Omega)$. Consequently, we have got a pair of densely-defined unbounded operators

$$\mathsf{d}_T : H_T\Lambda^k(\Omega) \subseteq L^2\Lambda^k(\Omega) \to L^2\Lambda^{k+1}(\Omega),$$
$$\delta_N : H_N\Lambda^k(\Omega) \subseteq L^2\Lambda^k(\Omega) \to L^2\Lambda^{k-1}(\Omega).$$

One can show that these two operators are closed, mutually adjoint, and have closed range. We can assemble a pair of mutually adjoint closed Hilbert complexes:

$$0 \longrightarrow H_T\Lambda^0 \subseteq L^2\Lambda^0 \xrightarrow{\mathsf{d}_T^0} \dots \xrightarrow{\mathsf{d}_T^{n-1}} H_T\Lambda^n \subseteq L^2\Lambda^n \longrightarrow 0, \tag{9}$$

$$0 \longleftarrow H_N^*\Lambda^0 \subseteq L^2\Lambda^0 \xleftarrow{\delta_N^1} \dots \xleftarrow{\delta_N^n} H_N^*\Lambda^n \subseteq L^2\Lambda^n \longleftarrow 0. \tag{10}$$

The second Hilbert complex is the adjoint of the first complex with respect to the L^2 product. The abstract theory of Hilbert complexes applies. In particular, we introduce the *k-th space of harmonic forms* $\mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N) := \ker \mathsf{d}_T^k \cap \ker \delta_N^k$.

The identity $b_k(\Omega, \Gamma_T) = \dim \mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N) = b_{n-k}(\Omega, \Gamma_N)$ (see [14, Theorem 5.3]) relates the dimension of the harmonic forms to the topological Betti numbers of the domain, and thus to the simplicial Betti numbers of any suitable triangulation.

We let μ_{Ω,Γ_N} denote the Poincaré-Friedrichs constant of the L^2 de Rham complex with partial boundary conditions. The square of μ_{Ω,Γ_N} bounds the operator norm of the generalized solution operator of the Hodge Laplace equation associated to that Hilbert complex; see [14]. We remark that the existence of a finite Poincaré-Friedrichs constant is typically shown using a compact embedding argument, and refer to [26], [21], [22], [25], [17] for further reading.

2.4. **Discrete Distributional Differential Forms.** Next, we set up the theory of discrete distributional differential forms. We assume that for each simplex $C \in \mathcal{T}$ and index $0 \leq k \leq \dim C$ we have fixed a finite-dimensional space $\Lambda^k(C)$ of smooth differential forms. We assume that $\operatorname{tr}^k_{C,F} \Lambda^k(C) = \Lambda^k(F)$ holds for $F \trianglelefteq C$, and moreover that $\mathsf{d}_C \Lambda^k(C) \subseteq \Lambda^{k+1}(C)$, so we have differential complexes

$$0 \to \Lambda^0(C) \xrightarrow{\mathsf{d}^0_C} \Lambda^1(C) \xrightarrow{\mathsf{d}^1_C} \dots \xrightarrow{\mathsf{d}^{m-1}_C} \Lambda^{\dim C}(C) \to 0. \tag{11}$$

We subsequently introduce $\Lambda^k_{-1}(\mathcal{T}^m, \mathcal{U}) := \bigoplus_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} \Lambda^k(C)$. We agree, letting $\mathcal{U}$ be understood, to abbreviate $\Lambda^k_{-1}(\mathcal{T}^m) \equiv \Lambda^k_{-1}(\mathcal{T}^m, \mathcal{U})$ in the sequel. The horizontal differential operators $\mathsf{D}^m_k : \Lambda^k_{-1}(\mathcal{T}^m) \longrightarrow \Lambda^{k+1}_{-1}(\mathcal{T}^m)$ are defined by

$$\mathsf{D}^m_k \omega := \sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} \mathsf{d}^k_C \omega_C, \quad \omega = (\omega_C)_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} \in \Lambda^k(\mathcal{T}^m).$$

The vertical differential operators $\mathsf{T}^m_k : \Lambda^k_{-1}(\mathcal{T}^m) \longrightarrow \Lambda^k_{-1}(\mathcal{T}^{m-1})$ are defined by

$$\mathsf{T}^m_k \omega := \sum_{\substack{C \in \mathcal{T}^m \setminus \mathcal{U}^m \\ F \in \mathcal{T}^{m-1} \setminus \mathcal{U}^{m-1} \\ F \trianglelefteq C}} o(F, C) \operatorname{tr}^k_{C,F} \omega_C, \quad \omega = (\omega_C)_{C \in \mathcal{T}^m - \mathcal{U}^m} \in \Lambda^k(\mathcal{T}^m).$$

One easily verifies $\mathsf{D}^m_{k+1} \mathsf{D}^m_k = 0$ and $\mathsf{T}^{m-1}_k \mathsf{T}^m_k = 0$. Furthermore, $\mathsf{D}^{m-1}_k \mathsf{T}^m_k = \mathsf{T}^m_{k+1} \mathsf{D}^m_k$. Hence we have a double complex in the sense of Gelfand and Manin [13].

$$\begin{tikzcd}
\Lambda^0_{-1}(\mathcal{T}^n) \arrow[r, "\mathsf{D}^n_0"] \arrow[d, "-\mathsf{T}^n_0"'] & \Lambda^1_{-1}(\mathcal{T}^n) \arrow[r, "\mathsf{D}^n_1"] \arrow[d, "-\mathsf{T}^n_1"'] & \Lambda^2_{-1}(\mathcal{T}^n) \arrow[r, "\mathsf{D}^n_2"] \arrow[d, "-\mathsf{T}^n_2"'] & \dots \\
\Lambda^0_{-1}(\mathcal{T}^{n-1}) \arrow[r, "-\mathsf{D}^{n-1}_0"] \arrow[d, "\mathsf{T}^{n-1}_0"'] & \Lambda^1_{-1}(\mathcal{T}^{n-1}) \arrow[r, "-\mathsf{D}^{n-1}_1"] \arrow[d, "\mathsf{T}^{n-1}_1"'] & \Lambda^2_{-1}(\mathcal{T}^{n-1}) \arrow[r, "-\mathsf{D}^{n-1}_2"] \arrow[d, "\mathsf{T}^{n-1}_2"'] & \dots \\
\Lambda^0_{-1}(\mathcal{T}^{n-2}) \arrow[r, "\mathsf{D}^{n-2}_0"] \arrow[d, "-\mathsf{T}^{n-2}_0"'] & \Lambda^1_{-1}(\mathcal{T}^{n-2}) \arrow[r, "\mathsf{D}^{n-2}_1"] \arrow[d, "-\mathsf{T}^{n-2}_1"'] & \Lambda^2_{-1}(\mathcal{T}^{n-2}) \arrow[r, "\mathsf{D}^{n-2}_2"] \arrow[d, "-\mathsf{T}^{n-2}_2"'] & \dots \\
\dots & \dots & \dots &
\end{tikzcd} \tag{12}$$

Its rows and columns are differential complexes on their own. We can derive conditions for the rows and columns to be exact sequences.

First we consider the *row complexes*

$$0 \to \Lambda^0_{-1}(\mathcal{T}^m) \xrightarrow{\mathsf{D}^m_0} \Lambda^1_{-1}(\mathcal{T}^m) \xrightarrow{\mathsf{D}^m_1} \dots \xrightarrow{\mathsf{D}^m_{m-1}} \Lambda^m_{-1}(\mathcal{T}^m) \to 0,$$

which are simply the direct sums of the simplex-wise sequences

$$0 \to \Lambda^0(C) \xrightarrow{\mathsf{d}^0_C} \Lambda^1(C) \xrightarrow{\mathsf{d}^1_C} \dots \xrightarrow{\mathsf{d}^{m-1}_C} \Lambda^m(C) \to 0.$$

We say that the *local exactness condition* holds if $\Lambda^0(C)$ contains the constant functions over C, and if additionally $\operatorname{ran} \mathsf{d}^{k-1} = \ker \mathsf{d}^k$ for $k \geq 1$. If the local exactness conditions hold, then $\ker \mathsf{D}_0^m$ is isomorphic to $\mathcal{C}_m(\mathcal{T},\mathcal{U})$.

Next we study the *column complexes*

$$\Lambda^k_{-1}(\mathcal{T}^n) \xrightarrow{\mathsf{T}^n_k} \Lambda^k_{-1}(\mathcal{T}^{n-1}) \xrightarrow{\mathsf{T}^{n-1}_k} \dots \xrightarrow{\mathsf{T}^{k+1}_k} \Lambda^k_{-1}(\mathcal{T}^k).$$

Similar to the treatment of the row complexes, the idea is to decompose the column complexes into "local" complexes. The implementation of that idea is, however, more technical in this case. As a technical definition, for $F \in \mathcal{T}$ we let $\mathring{\Lambda}^k(C)$ denote the subspace of $\Lambda^k(C)$ whose members have vanishing traces on the boundary simplices of F. Two conditions together imply exactness of the column complexes.

We say that the *geometric decomposition condition* holds if we have linear extension operators $\operatorname{ext}^k_{F,C} : \mathring{\Lambda}^k(F) \to \Lambda^k(C)$ for $C \in \mathcal{T}$ with $F \trianglelefteq C$, that satisfy the following conditions: we have $\operatorname{tr}^k_{C,F} \operatorname{ext}^k_{F,C} = \operatorname{Id}_{\mathring{\Lambda}^k(F)}$, we have $\operatorname{tr}^k_{C,G} \operatorname{ext}^k_{F,C} = \operatorname{ext}^k_{F,G}$ for all $F \trianglelefteq G \trianglelefteq C$, and we have $\operatorname{tr}^k_{C,G} \operatorname{ext}^k_{F,C} = 0$ for $G \trianglelefteq C$ and $F \ntrianglelefteq G$. Under the geometric decomposition assumption, we have a direct decomposition of $\Lambda^k_{-1}(\mathcal{T}^m)$ into subspaces

$$\Lambda^k_{-1}(\mathcal{T}^m) = \bigoplus_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} \bigoplus_{F \trianglelefteq C} \operatorname{ext}^k_{F,C} \mathring{\Lambda}^k(F). \tag{13}$$

Example 2.1. *The spaces $\mathcal{P}_r\Lambda^k(C)$ and $\mathcal{P}^-_r\Lambda^k(C)$ satisfy the geometric decomposition assumption. This was proven in Section 4 of* [1], *and we refer to* [2] *for an elaboration of the details. This holds more generally for compatible finite element systems* [8].

Let us henceforth assume that the geometric decomposition condition is valid. By reordering the previous direct sum, we obtain the desired decomposition of the column complexes. We have

$$\Lambda^k_{-1}(\mathcal{T}^m) = \bigoplus_{F \in \mathcal{T}} \Gamma^m_k(F), \quad \Gamma^m_k(F) := \bigoplus_{\substack{C \in \mathcal{T}^m \setminus \mathcal{U}^m \\ F \trianglelefteq C}} \operatorname{ext}^k_{F,C} \mathring{\Lambda}^k(F), \quad F \in \mathcal{T},$$

Hence the columns are the direct sums of the "local" vertical sequences

$$0 \to \Gamma^n_k(F) \cap \ker \mathsf{T}^n_k \longrightarrow \Gamma^n_k(F) \xrightarrow{\mathsf{T}^n_k} \dots \xrightarrow{\mathsf{T}^{k+1}_k} \Gamma^k_k(F) \to 0$$

In order to obtain a useful result on their homology, we introduce a further condition on the triangulation. For $F \in \mathcal{T}$ we define simplicial complexes

$$\mathcal{M}_F := \{ G \in \mathcal{T} \mid \exists C \in \mathcal{T}^n : F \trianglelefteq C \text{ and } G \trianglelefteq C \}, \tag{14a}$$

$$\mathcal{N}_F := \left\{ G \in \mathcal{M}^{[n-1]}_F \mid F \ntrianglelefteq G \text{ or } G \in \mathcal{U} \right\}. \tag{14b}$$

The simplicial complex $\mathcal{M}_F$ describes the local patch of simplices in $\mathcal{T}$ adjacent to F, whereas the subcomplex $\mathcal{N}_F \subseteq \mathcal{M}_F$ describes the boundary of that patch. We say that $\mathcal{T}$ satisfies the *local patch condition* relative to $\mathcal{U}$ if

$$0 \to \mathcal{C}_n(\mathcal{M}_F,\mathcal{N}_F) \xrightarrow{\partial^n} \dots \xrightarrow{\partial^1} \mathcal{C}_0(\mathcal{M}_F,\mathcal{N}_F) \to 0 \tag{15}$$

with vanishing homology spaces at indices $n-1,\dots,0$, for all $F \in \mathcal{T}$. Equivalently, $b_l(\mathcal{M}_F,\mathcal{N}_F) = 0$ for all $0 \leq l \leq n-1$, that is, all simplicial Betti numbers except for the top-dimensional one vanish. If the geometric decomposition condition and the local patch condition hold, then it can be shown [19, Lemma 3] that the differential complex (15) has vanishing cohomology spaces at the indices $0 \leq m \leq n-1$.

Example 2.2. *These assumptions are motivated by the differential complexes of finite element exterior calculus. On each simplex we may fix a differential complex by choosing* $\Lambda^k(C) = \mathcal{P}_r\Lambda^k(C)$ *or* $\Lambda^k(C) = \mathcal{P}_r^-\Lambda^k(C)$ *of* [1]*, provided that* $\mathsf{d}_C^k\Lambda^k(C) \subseteq \Lambda^k(C)$*. If* $\operatorname{tr}_{C,F}^k \Lambda^k(C) = \Lambda^{k+1}(F)$ *holds, then the basic setting of this section applies. More general assumptions have been discussed within the framework of* element systems *in* [7]*, where the trace operators are not assumed onto.*

It was proven in Section 3 of [1] *that the local exactness condition holds if whenever* $\Lambda^k(C) = \mathcal{P}_r\Lambda^k(C)$ *or* $\Lambda^k(C) = \mathcal{P}_r^-\Lambda^k(C)$ *we already have* $\Lambda^{k+1}(C) = \mathcal{P}_{r-1}\Lambda^{k+1}(C)$ *or* $\Lambda^{k+1}(C) = \mathcal{P}_r^-\Lambda^{k+1}(C)$*. The exactness of finite element differential complexes was also used in* [7, (5.3)]*. The local patch condition is satisfied when* $\mathcal{T}$ *triangulates a manifold with boundary, and we refer to* [19] *for further details.*

2.5. **Scalar Products.** In this section we assume that each $\Lambda^k_{-1}(\mathcal{T}^m)$ is equipped with a Hilbert space structure. A Hilbert space structure is uniquely determined by a scalar product $\langle \cdot, \cdot \rangle$, and we review some possible choices of scalar products.

Generalizing a scalar product that was used in [5], we may consider

$$\langle \omega, \eta \rangle_h := \sum_{C \in \mathcal{T}^m} h_C^{n-m} \langle \omega_C, \eta_C \rangle_{L^2\Lambda^k(C)}, \quad \omega, \eta \in \Lambda^k_{-1}(\mathcal{T}^m). \tag{16}$$

Here, h_C denotes the diameter of a simplex $C \in \mathcal{T}$; we refer to Section 3 for more details. In later sections of this article, however, we focus on the scalar product

$$\langle \omega, \eta \rangle_{-h} := \sum_{C \in \mathcal{T}^m} h_C^{m-n} \langle \omega_C, \eta_C \rangle_{L^2\Lambda^k(C)}, \quad \omega, \eta \in \Lambda^k(\mathcal{T}^m). \tag{17}$$

In Sections 5 and 6 we prove Poincaré-Friedrichs inequalities with respect to (17).

2.6. **Finite Element Differential Complexes.** We are particularly interested in the kernel spaces of the vertical and horizontal differential operators. We introduce

$$\Lambda^k(\mathcal{T}^m) := \Lambda^k_{-1}(\mathcal{T}^m) \cap \ker \mathsf{T}_k^m, \quad \Gamma^k(\mathcal{T}^m) := \Lambda^k_{-1}(\mathcal{T}^m) \cap \ker \mathsf{D}_k^m. \tag{18}$$

Writing $\Lambda^k_0(\mathcal{T}^m) \equiv \Lambda^k(\mathcal{T}^m)$ and $\Gamma^k_0(\mathcal{T}^m) \equiv \Gamma^k(\mathcal{T}^m)$ will be helpful in some of the subsequent proofs. The inclusions $\mathsf{D}_k^m\Lambda^k(\mathcal{T}^m) \subseteq \Lambda^{k+1}(\mathcal{T}^m)$ and $\mathsf{T}_k^m\Lambda^k(\mathcal{T}^m) \subseteq \Lambda^k(\mathcal{T}^{m-1})$ follow from definitions. Consequently, we have differential complexes

$$\Lambda^0(\mathcal{T}^m) \xrightarrow{\mathsf{D}_0^m} \Lambda^1(\mathcal{T}^m) \xrightarrow{\mathsf{D}_1^m} \dots \xrightarrow{\mathsf{D}_{m-1}^m} \Lambda^m(\mathcal{T}^m), \tag{19}$$

$$\Gamma^k(\mathcal{T}^n) \xrightarrow{\mathsf{T}_k^n} \Gamma^k(\mathcal{T}^{n-1}) \xrightarrow{\mathsf{T}_k^{n-1}} \dots \xrightarrow{\mathsf{T}_k^1} \Gamma^k(\mathcal{T}^k). \tag{20}$$

Given differential complexes, it is natural to study their homology spaces. With an additional Hilbert structure, we more precisely determine the harmonic spaces of Hilbert complexes. We introduce the spaces

$$\mathfrak{H}^k(\mathcal{T}^m) := \left\{ \omega \in \Lambda^k(\mathcal{T}^m) \;\middle|\; \begin{array}{l} \omega \in \ker \mathsf{D}_k^m, \\ \forall \eta \in \mathsf{D}_{k-1}^m\Lambda^k(\mathcal{T}^m) : \langle \omega, \eta \rangle = 0 \end{array} \right\}, \tag{21}$$

$$\mathfrak{C}^k(\mathcal{T}^m) := \left\{ \omega \in \Gamma^k(\mathcal{T}^m) \;\middle|\; \begin{array}{l} \omega \in \ker \mathsf{T}_k^m, \\ \forall \eta \in \mathsf{T}_k^{m+1}\Gamma^k(\mathcal{T}^{m+1}) : \langle \omega, \eta \rangle = 0 \end{array} \right\}. \tag{22}$$

These spaces play the role of discrete harmonic spaces in generalized finite element complexes. The following statement has been shown in [19].

Proposition 2.3. *Suppose that the local exactness condition, the geometric decomposition condition, and the local patch condition hold. Then we have got isomorphisms*

$$\mathcal{H}_{n-k}(\mathcal{T},\mathcal{U}) \simeq \mathfrak{C}^0(\mathcal{T}^{n-k}) \simeq \mathfrak{H}^k(\mathcal{T}^n).$$

Remark 2.4. *On the one hand, the space $\Lambda^k(\mathcal{T}^n)$ is a subspace of $L^2\Lambda^k(\Omega)$. Any $\omega \in \Lambda^k(\mathcal{T}^n)$ restricts to a differential form in $\Lambda^k(T)$ on each n-simplex $T \in \mathcal{T}$. The restrictions of ω to cells $T, T' \in \mathcal{T}$ which share a common face F have the same trace on F. Moreover, ω has vanishing trace along $\mathcal{V}$. It follows that the differential complex* (19) *is a subcomplex of* (9). *On the other hand, the space $\Gamma^0(\mathcal{T}^m)$ is isomorphic to the space of chains $\mathcal{C}_m(\mathcal{T},\mathcal{U})$. Proposition* (2.3) *states that the complexes* (19) *and* (20) *have homology spaces of the same dimension.*

2.7. **Distributional complexes.** Whereas these definitions and results are of interest on their own, the major motivation for this research has been to generalize the distributional finite element complexes in [5], which study the construction of equilibrated a posteriori error estimators. In order to generalize these differential complexes, we introduce complexes of *discrete distributional differential forms.* We start with further definitions and let

$$\Lambda^k_{-b}(\mathcal{T}^m) := \bigoplus_{i=0}^{b-1} \Lambda^{k-i}_{-1}(\mathcal{T}^{m-i}), \quad 0 \leq b \leq m+1, \tag{23}$$

$$\Gamma^k_{-b}(\mathcal{T}^m) := \bigoplus_{i=0}^{b-1} \Lambda^{k+i}_{-1}(\mathcal{T}^{m+i}), \quad 0 \leq b \leq n-m+1. \tag{24}$$

At this point we notice that $\Lambda^m_0(\mathcal{T}^m) = \Lambda^m_{-1}(\mathcal{T}^m) = \Gamma^m_{-1}(\mathcal{T}^m) = \Gamma^m_0(\mathcal{T}^m)$. We generalize the exterior derivative to act on $\Lambda^k_{-b}(\mathcal{T}^m)$ and $\Gamma^k_{-b}(\mathcal{T}^m)$. We define

$$\mathsf{d}^k : \Lambda^{k-i}_{-1}(\mathcal{T}^{m-i}) \to \Lambda^{k-i+1}_{-1}(\mathcal{T}^{m-i}) \oplus \Lambda^{k-i}_{-1}(\mathcal{T}^{m-i-1}) \tag{25}$$

by setting

$$\mathsf{d}^k\omega := (-1)^{n-m}\mathsf{D}^m_k\omega + (-1)^{n-m+1}\mathsf{T}^m_k\omega, \quad \omega \in \Lambda^k_{-1}(\mathcal{T}^m). \tag{26}$$

The definition immediately yields the differential property $\mathsf{d}^{k+1}\mathsf{d}^k\omega = 0$, valid for all $\omega \in \Lambda^{k-i}_{-1}(\mathcal{T}^{m-i})$, $0 \leq i \leq k$. With this definition we naturally obtain mappings

$$\mathsf{d}^{k-n+m} : \Lambda^k_{-b}(\mathcal{T}^m) \to \Lambda^k_{-b}(\mathcal{T}^m), \quad \mathsf{d}^{k-n+m} : \Gamma^k_{-b}(\mathcal{T}^m) \to \Gamma^k_{-b}(\mathcal{T}^m), \tag{27}$$

$$\mathsf{d}^{k-n+m} : \Lambda^k(\mathcal{T}^m) \to \Lambda^{k+1}(\mathcal{T}^m), \quad \mathsf{d}^{k-n+m} : \Gamma^k(\mathcal{T}^m) \to \Gamma^k(\mathcal{T}^{m-1}). \tag{28}$$

The elements of the spaces $\Lambda^k_{-b}(\mathcal{T}^m)$ and $\Gamma^k_{-b}(\mathcal{T}^m)$ are called *discrete distributional differential forms*, and the extension of the exterior derivative to these spaces is called *discrete distributional exterior derivative.*

Remark 2.5. *At this point we justify the terminology* discrete distributional differential form. *Let $C \in \mathcal{T}^m$ and let $\omega_C \in \Lambda^k(C) \subseteq \Lambda^k_{-1}(\mathcal{T}^m)$. If $\phi \in C^\infty\Lambda^{n-m+k}_T(\Omega)$, then $\operatorname{tr}_C \star_\Omega \phi \in C^\infty\Lambda^{m-k}_T(\Omega)$. We interpret ω_C as a linear functional on $C^\infty\Lambda^{n-m+k}_T(\Omega)$*

by the association $\phi \mapsto \int_C \omega_C \wedge \operatorname{tr}_C \star_\Omega \phi$. *If now* $\psi \in C^\infty \Lambda^{n-m+k+1}_T(\Omega)$, *then integration by parts shows*

$$\int_C \omega_C \wedge \operatorname{tr}_C \star_\Omega \delta^{n-m+k+1}_\Omega \psi = (-1)^{n-m} \int_C \mathsf{d}^k_C \omega_C \wedge \operatorname{tr}_C \star_\Omega \psi \\ + (-1)^{n-m+1} \sum_{\substack{C \in \mathcal{T}^m \setminus \mathcal{U}^m \\ F \in \mathcal{T}^{m-1} \setminus \mathcal{U}^{m-1} \\ F \trianglelefteq C}} o(F,C) \int_F \operatorname{tr}^k_{C,F} \omega_C \wedge \operatorname{tr}_F \star_\Omega \psi.$$

This motivates the definition of the operators D^m_k *and* T^m_k, *and subsequently the definition of the discrete distributional exterior derivative. This is also reflected in the sign convention of Diagram* (12).

We obtain several Hilbert complexes by appending mappings of the form (27) and (28) such that the source and target spaces of successive arrows match. To begin with, it is easy to see that we have well-defined Hilbert complexes

$$0 \to \Lambda^0(\mathcal{T}^n) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \Lambda^n(\mathcal{T}^n) \to 0, \tag{29}$$

$$0 \to \Gamma^0(\mathcal{T}^n) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \Gamma^0(\mathcal{T}^0) \to 0. \tag{30}$$

The first complexes resembles the classical finite element complex, while the second complex resembles the simplicial chain complex. At any index, we may redirect these complexes. At a fixed index k, we may change the first complex to

$$\dots \xrightarrow{\mathsf{d}^{k-2}} \Lambda^{k-1}(\mathcal{T}^n) \xrightarrow{\mathsf{d}^{k-1}} \Lambda^k_{-1}(\mathcal{T}^n) \xrightarrow{\mathsf{d}^{k}} \Lambda^{k+1}_{-2}(\mathcal{T}^n) \xrightarrow{\mathsf{d}^{k+1}} \dots \tag{31}$$

and, at any index k, we may redirect the second complex to

$$\dots \xrightarrow{\mathsf{d}^{k-2}} \Gamma^0(\mathcal{T}^{n-k+1}) \xrightarrow{\mathsf{d}^{k-1}} \Gamma^0_{-1}(\mathcal{T}^{n-k}) \xrightarrow{\mathsf{d}^{k}} \Gamma^0_{-2}(\mathcal{T}^{n-k-1}) \xrightarrow{\mathsf{d}^{k+1}} \dots \tag{32}$$

We see that (29) and (30) are already trivially redirected at the last index. In this manner, we derive two families of Hilbert complexes, each consisting of the Hilbert complexes redirected at one of the $n+1$ indices. We moreover see that within each family, the Hilbert complexes redirected at later indices are contained in the Hilbert complexes redirected at earlier indices. One can use this observation to relate the harmonic spaces of the Hilbert complexes in each family.

With regard to this inclusion ordering, each family contains maximal complexes

$$0 \to \Lambda^0_{-1}(\mathcal{T}^n) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \Lambda^n_{-n-1}(\mathcal{T}^n) \to 0, \tag{33}$$

$$0 \to \Gamma^0_{-1}(\mathcal{T}^n) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \Gamma^0_{-n-1}(\mathcal{T}^0) \to 0. \tag{34}$$

We see that (33) and (34) are, in fact, identical. With that easy observation in mind, we put the harmonic spaces of these Hilbert complexes into relation.

Recall that the harmonic spaces of the Hilbert complexes (29) – (34), whose members we call *discrete distributional harmonic forms*, are defined as

$$\mathfrak{H}^k_{-b}(\mathcal{T}^m) := \left\{ \omega \in \Lambda^k_{-b}(\mathcal{T}^m) \;\middle|\; \begin{array}{l} \mathsf{d}^{k+n-m}\omega = 0, \\ \forall \eta \in \mathsf{d}^{k+n-m-1}\Lambda^{k-1}_{-b+1}(\mathcal{T}^m) : \langle \omega, \eta \rangle = 0 \end{array} \right\}, \tag{35}$$

$$\mathfrak{C}^k_{-b}(\mathcal{T}^m) := \left\{ \omega \in \Gamma^k_{-b}(\mathcal{T}^m) \;\middle|\; \begin{array}{l} \mathsf{d}^{k+n-m}\omega = 0, \\ \forall \eta \in \mathsf{d}^{k+n-m-1}\Gamma^{k}_{-b+1}(\mathcal{T}^{m+1}) : \langle \omega, \eta \rangle = 0 \end{array} \right\}. \tag{36}$$

where $b \geq 0$. We may abbreviate $\mathfrak{H}^k_0(\mathcal{T}^m) \equiv \mathfrak{H}^k(\mathcal{T}^m)$ and $\mathfrak{C}^k_0(\mathcal{T}^m) \equiv \mathfrak{C}^k(\mathcal{T}^m)$ for notational convenience.

One can derive isomorphisms between the spaces $\mathfrak{H}^{k}_{-b}(\mathcal{T}^m)$ and $\mathfrak{C}^{k}_{-b}(\mathcal{T}^m)$ when $b \geq 0$ varies. For this, we assume that we have fixed mappings

$$\mathsf{E}^m_k : \Lambda^k_{-1}(\mathcal{T}^{m-1}) \to \Lambda^k_{-1}(\mathcal{T}^m), \quad \mathsf{P}^m_k : \Lambda^{k+1}_{-1}(\mathcal{T}^m) \to \Lambda^k_{-1}(\mathcal{T}^m)$$

which satisfy

$$\mathsf{T}^m_k = \mathsf{T}^m_k \mathsf{E}^m_k \mathsf{T}^m_k, \quad \mathsf{D}^m_k = \mathsf{D}^m_k \mathsf{P}^m_k \mathsf{D}^m_k.$$

The operators E^m_k and P^m_k are generalized inverses of T^m_k and D^m_k, respectively. With such operators fixed, we then define

$$R_{k,b} : \Lambda^k_{-b}(\mathcal{T}^n) \to \Lambda^k_{-b}(\mathcal{T}^n), \quad \omega \mapsto \omega + (-1)^b \mathsf{d}^{k-1} \mathsf{E}^{n-b+2}_{k-b+1} \omega, \tag{37}$$

$$S_{m,b} : \Gamma^0_{-b}(\mathcal{T}^m) \to \Gamma^0_{-b}(\mathcal{T}^m), \quad \omega \mapsto \omega + (-1)^{b+n-m} \mathsf{d}^{k+n-m-1} \mathsf{P}^{m+b-1}_{b-2} \omega. \tag{38}$$

As already mentioned in [19], the Moore-Penrose pseudoinverses ([11]) of D^m_k and T^m_k are a possible choice for P^m_k and E^m_k. In Section 4 we will introduce generalized inverses with stronger properties, but those are not needed at this point.

We can now derive relations between spaces of discrete distributional harmonic forms. We summarize the results of [19] in this regard. We first provide technical observations that pertain to the range of the discrete distributional exterior derivative:

Lemma 2.6. *Suppose that $b > 1$. If $\omega \in \Lambda^k_{-b}(\mathcal{T}^n)$ with $\mathsf{d}^k \omega \in \Lambda^{k+1}_{-b}(\mathcal{T}^n)$, then*

$$\mathsf{d}^k R_{k,b} \omega = \mathsf{d}^k \omega, \quad R_{k,b} \omega \in \Lambda^k_{-b+1}(\mathcal{T}^n).$$

If $\omega \in \Gamma^0_{-b}(\mathcal{T}^n)$ with $\mathsf{d}^{n-m} \omega \in \Gamma^0_{-b}(\mathcal{T}^{m-1})$, then

$$\mathsf{d}^{n-m} S_{m,b} \omega = \mathsf{d}^{n-m} \omega, \quad S_{m,b} \omega \in \Gamma^0_{-b+1}(\mathcal{T}^m).$$

Lemma 2.7. *Suppose that $b \geq 0$. If $\omega \in \Lambda^k_{-b+1}(\mathcal{T}^n)$ with $\mathsf{d}^k \omega = 0$, then ω is not orthogonal to $\mathsf{d}^{k-1} \Lambda^{k-1}_{-b+1}(\mathcal{T}^n)$. If $\omega \in \Gamma^0_{-b+1}(\mathcal{T}^m)$ with $\mathsf{d}^{n-m} \omega = 0$, then ω is not orthogonal to $\mathsf{d}^{n-m-1} \Gamma^0_{-b+1}(\mathcal{T}^{m+1})$.*

The discrete distributional harmonic spaces $\mathfrak{H}^k_{-1}(\mathcal{T}^n)$ and $\mathfrak{C}^0_{-1}(\mathcal{T}^m)$ are easy to describe. The spaces $\mathfrak{H}^k_{-b}(\mathcal{T}^n)$ and $\mathfrak{C}^0_{-b}(\mathcal{T}^m)$ for general $b \geq 2$ can be derived recursively.

Lemma 2.8. *We have $\mathfrak{H}^k(\mathcal{T}^n) = \mathfrak{H}^k_{-1}(\mathcal{T}^n)$ and $\mathfrak{C}^0(\mathcal{T}^m) = \mathfrak{C}^0_{-1}(\mathcal{T}^m)$.*

Lemma 2.9. *Suppose that $b \geq 2$.*

Let $P_{\ker \mathsf{d}^k}$ denote the orthogonal projection onto the kernel of the operator $\mathsf{d}^k : \Lambda^k_{-b}(\mathcal{T}^n) \to \Lambda^{k+1}_{-b+1}(\mathcal{T}^n)$. Then the operator $P_{\ker \mathsf{d}^k} R^*_{k,b}$ acts as an isomorphism from $\mathfrak{H}^k_{-b+1}(\mathcal{T}^n)$ to $\mathfrak{H}^k_{-b}(\mathcal{T}^n)$.

Let $P_{\ker \mathsf{d}^{n-m}}$ denote the orthogonal projection onto the kernel of the operator $\mathsf{d}^{n-m} : \Gamma^0_{-b}(\mathcal{T}^m) \to \Gamma^0_{-b+1}(\mathcal{T}^{m-1})$. Then the operator $P_{\ker \mathsf{d}^{n-m}} S^*_{m,b}$ acts as an isomorphism from $\mathfrak{C}^0_{-b+1}(\mathcal{T}^m)$ to $\mathfrak{C}^0_{-b}(\mathcal{T}^m)$.

These findings culminate into the following main result of [19].

Theorem 2.10. *Under the assumptions of this section, we have isomorphisms between harmonic spaces:*

$$\begin{aligned}\mathcal{H}^{n-k}(\mathcal{T}, \mathcal{U}) \simeq \mathfrak{C}^0(\mathcal{T}^{n-k}) &= \mathfrak{C}^0_{-1}(\mathcal{T}^{n-k}) \simeq \cdots \simeq \mathfrak{C}^0_{-k-1}(\mathcal{T}^{n-k}) \\ &= \mathfrak{H}^k_{-k-1}(\mathcal{T}^n) \simeq \cdots \simeq \mathfrak{H}^k_{-1}(\mathcal{T}^n) = \mathfrak{H}^k(\mathcal{T}^n).\end{aligned}$$

3. Geometric Regularity

In this section, we review notions of regularity for triangulations.

Let h_C be the diameter of any simplex $C \in \mathcal{T}$ and let $\mathrm{vol}^m(C)$ be its m-dimensional volume if C is m-dimensional. When C is a vertex, then we define h_C as the minimum diameter of all simplices adjacent to C and we have $\mathrm{vol}^0(C) = 1$ by definition. There exists a minimal constant $\mu_{\mathcal{T}} > 0$, called the *shape-constant* of $\mathcal{T}$, for which

$$\forall 0 \leq m \leq n : \forall C \in \mathcal{T} : h_C^m \leq \mu_{\mathcal{T}} \,\mathrm{vol}^m(C), \tag{39}$$

$$\forall C', C \in \mathcal{T}, C \cap C' \neq \emptyset : h_{C'} \leq \mu_{\mathcal{T}} h_C. \tag{40}$$

The shape-constant quantifies how far a simplex $C \in \mathcal{T}$ is from being degenerate and how comparable adjacent simplices are in size. In applications, we handle families of triangulations whose shape-constants are uniformly bounded. This is the case for many algorithmically refined sequences of triangulations. Our notion of shape-constant captures what is usually called *shape-regularity.*

We define the *m-dimensional reference simplex* Δ^m as the convex closure of the origin and the coordinate vectors in $\mathbb{R}^m$, that is, $\Delta^m := \mathrm{convex}\,\{0, e_1, \dots, e_m\} \subset \mathbb{R}^m$. For any m-simplex $C \in \mathcal{T}^m$ we fix an affine mapping $\Phi_C : \Delta^m \to C$ that maps the m-dimensional reference simplex bijectively onto C.

In addition to reference simplices of various dimensions, we introduce reference patches. For each local patch $\mathcal{M}_F$, $F \in \mathcal{T}$, we fix a reference patch Δ_F whose construction we outline as follows: Δ_F is constructed by gluing n-dimensional reference cells such that the resulting simplicial complex is combinatorially isomorphic to $\mathcal{M}_F$. This means in particular that we can fix a piecewise affine topological homeomorphism $\Psi_F : \Delta_F \to \mathcal{M}_F$.

There exists a constant $\mu_N > 0$, only depending on the shape-constant $\mu_{\mathcal{T}}$, such that any simplex $C \in \mathcal{T}$ is adjacent to at most μ_N simplices from $\mathcal{T}$. In particular, this bounds the number of possible combinatorial structures for the local patches $\mathcal{M}_F$ as $F \in \mathcal{T}$ varies. Thus, without loss of generality the reference patches Δ_F, $F \in \mathcal{T}$, constitute a finite set whose cardinality depends only on μ_N.

4. Local Estimates

In this section we give inequalities satisfied by the horizontal and vertical differential operators D_k^m and T_k^m and introduce specific choices for the generalized inverses P_k^m and E_k^m, which have been introduced in Section 2 in a general form.

Below, we make the following assumption on the finite element spaces for some $R \in \mathbb{N}_0$, which we call *polynomial degree R condition*: There exists $R \in \mathbb{N}_0$ such that for each $C \in \mathcal{T}$ we have $\mathring{\Lambda}^k(C) \subseteq \mathring{\mathcal{P}}_R\Lambda^k(C)$.

We first consider the horizontal differential operator and a special choice of generalized inverse. We leverage on the existence of an operator

$$\widehat{\mathsf{P}}_k^m : L^2\Lambda^{k+1}(\Delta^m) \to H\Lambda^k(\Delta^m)$$

such that $\widehat{\mathsf{P}}_k^m\big(\mathcal{P}_r\Lambda^{k+1}(\Delta^m)\big) \subseteq \mathcal{P}_r^-\Lambda^k(\Delta^m)$ and

$$\mathsf{d}_{\Delta^m}^k \widehat{\mathsf{P}}_k^m \mathsf{d}_{\Delta^m}^k \xi = \mathsf{d}_{\Delta^m}^k \xi, \quad \xi \in H\Lambda^k(\Delta^m).$$

The existence of such an operator $\widehat{\mathsf{P}}^m_k$ follows immediately from Proposition 4.2 of [10] and Lemma 3.8 of [1]. We then define

$$\mathsf{P}^m_k : \Lambda^{k+1}(\mathcal{T}^m) \to \Lambda^k(\mathcal{T}^m), \quad \sum_{C \in \mathcal{T}^m} \omega_C \mapsto \sum_{C \in \mathcal{T}^m} \Phi_C^{-*} \widehat{\mathsf{P}}^m_k \Phi_C^* \omega_C.$$

This is an admissible choice for a generalized inverse of the horizontal derivative as described in Section 2. We call P^m_k the *horizontal antiderivative*. We have

$$\mathsf{D}^m_k = \mathsf{D}^m_k \mathsf{P}^m_k \mathsf{D}^m_k.$$

Analogous constructions for the vertical differential operators require more technical definitions. First, by the geometric decomposition assumption, for each $\omega \in \Lambda^k(C)$ and $C \in \mathcal{T}$ we have the unique decomposition

$$\omega = \sum_{F \trianglelefteq C} \operatorname{ext}_{F,C} \zeta_C^F, \quad \zeta_C^F \in \mathring{\Lambda}^k(F).$$

We recall that, as a consequence of the geometric decomposition assumption, we have for each $\omega \in \Lambda^k(\mathcal{T}^m)$ a unique decomposition

$$\omega = \sum_{F \in \mathcal{T}^{[m]}} \zeta^F, \quad \zeta^F = \sum_{C \in \mathcal{T}^m} \operatorname{ext}_{F,C} \zeta_C^F.$$

The vertical differential operator T^m_k preserves the decomposition of ω into terms associated to simplices $F \in \mathcal{T}^{[m]}$. In particular, the vertical complex

$$\Lambda^k(\mathcal{T}^n) \xrightarrow{\mathsf{T}^n_k} \dots \xrightarrow{\mathsf{T}^{k+1}_k} \Lambda^k(\mathcal{T}^k)$$

is the direct sum of simplex-related differential complexes

$$\Gamma^n_k(F) \xrightarrow{\mathsf{T}^n_k} \dots \xrightarrow{\mathsf{T}^{k+1}_k} \Gamma^m_k(F), \quad F \in \mathcal{T}.$$

For each $F \in \mathcal{T}$, this simplex-related differential complex is isomorphic to

$$\Psi_F^* \Gamma^n_k(F) \xrightarrow{\Psi_F^* \mathsf{T}^n_k \Psi_F^{-*}} \dots \xrightarrow{\Psi_F^* \mathsf{T}^{k+1}_k \Psi_F^{-*}} \Psi_F^* \Gamma^m_k(F). \tag{41}$$

For each $F \in \mathcal{T}$, we fix an operator $\widehat{\mathsf{E}}^m_{k,F} : \Psi_F^* \Gamma^{m-1}_k(F) \to \Psi_F^* \Gamma^m_k(F)$ that satisfies $\mathsf{T}^m_k \Psi_F^{-*} \widehat{\mathsf{E}}^m_{k,F} \Psi_F^* \mathsf{T}^m_k \zeta^F = \mathsf{T}^m_k \zeta^F$ for all $\zeta^F \in \Gamma^m_k(F)$. We then define

$$\mathsf{E}^m_k : \Lambda^k(\mathcal{T}^{m-1}) \to \Lambda^k(\mathcal{T}^{m-1}), \quad \sum_{F \in \mathcal{T}^{[m-1]}} \zeta^F \to \sum_{F \in \mathcal{T}^{[m-1]}} \Psi_F^{-*} \widehat{\mathsf{E}}^m_{k,F} \Psi_F^* \zeta^F.$$

By construction we have the identity

$$\mathsf{T}^m_k \mathsf{E}^m_k \mathsf{T}^m_k \omega = \mathsf{T}^m_k \omega, \quad \omega \in \Lambda^k(\mathcal{T}^m).$$

By the results of Section 3, we may assume without loss of generality that the collection $(\mathsf{E}^m_{k,F})_{F \in \mathcal{T}}$ has a cardinality that can be bounded in terms of the shape constant.

With this preparation, the following estimates are known [20].

Lemma 4.1. *There exist constants $\check{\mu}, \hat{\mu} \geq 0$, depending only on R and μ_M, such that for all $\alpha \in \mathbb{R}$ and $\omega \in \Lambda^k_{-1}(\mathcal{T}^m)$ we have*

$$\sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} h_C^\alpha \left\| (\mathsf{D}^m_k \omega)_C \right\|^2_{L^2\Lambda^{k+1}(C)} \leq \check{\mu} \sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} h_C^{\alpha-2} \left\| \omega_C \right\|^2_{L^2\Lambda^k(C)}, \tag{42}$$

$$\sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} h_C^\alpha \left\| (\mathsf{P}^m_{k-1} \omega)_C \right\|^2_{L^2\Lambda^{k-1}(C)} \leq \hat{\mu} \sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} h_C^{\alpha+2} \left\| \omega_C \right\|^2_{L^2\Lambda^k(C)}. \tag{43}$$

Let $\alpha \in \mathbb{R}$. There exist constants $\grave{\mu}, \acute{\mu} \geq 0$, depending only on R, α and $\mu_{\mathcal{T}}$, such that for all $\omega \in \Lambda^k_{-1}(\mathcal{T}^m)$ we have

$$\sum_{Q \in \mathcal{T}^{m-1}} h_Q^\alpha \|(\mathsf{T}_k^m \omega)_Q\|^2_{L^2\Lambda^k(Q)} \leq \grave{\mu} \sum_{C \in \mathcal{T}^m} h_C^{\alpha-1} \|\omega_C\|^2_{L^2\Lambda^k(C)}, \tag{44}$$

$$\sum_{T \in \mathcal{T}^{m+1}} h_T^\alpha \|(\mathsf{E}_k^m \omega)_T\|^2_{L^2\Lambda^k(T)} \leq \acute{\mu} \sum_{C \in \mathcal{T}^m} h_C^{\alpha+1} \|\omega_C\|^2_{L^2\Lambda^k(C)}. \tag{45}$$

Remark 4.2. *The polynomial degree R condition holds, of course, for any fixed finite element de Rham complex. In practice, we typically consider families of such complexes that uniformly satisfy the polynomial degree R condition.*

5. Hilbert Chain Complexes

Given a triangulation, the duality between the corresponding differential complex of Whitney forms and the corresponding differential complex of simplicial chains is a well-known fact of differential topology. In this section we use this duality to derive Poincaré-Friedrichs-type inequalities on simplicial chains with respect to mesh-dependent norms.

We assume that the reader is familiar with the notion of lowest-order Whitney m-forms over simplices. We denote that space by $\mathcal{P}_1^-\Lambda^m(C)$, and refer to Section 3–4 of [1] for definitions and further details; an important fact is that

$$\mathsf{d}^m \mathcal{P}_1^-\Lambda^m(C) \subseteq \mathcal{P}_1^-\Lambda^{m+1}(C).$$

We let $\mathcal{W}^m(\mathcal{T})$, the space of *Whitney m-forms over $\mathcal{T}$*, be the vector space of collections $(\phi_C)_{C\in\mathcal{T}}$ of smooth differential m-forms over the simplices of $\mathcal{T}$ that satisfy

$$(\phi_C)_{C\in\mathcal{T}} \in \bigoplus_{C\in\mathcal{T}} \mathcal{P}_1^-\Lambda^m(C), \tag{46}$$

$$\forall C \in \mathcal{T}, F \trianglelefteq C : \operatorname{tr}_{C,F} \phi_C = \phi_F. \tag{47}$$

We generally consider Whitney forms with partial boundary conditions. We define

$$\mathcal{W}^m(\mathcal{T},\mathcal{U}) := \left\{ \phi \in \mathcal{W}^m(\mathcal{T}) \middle| \forall C \in \mathcal{U} : \phi_C = 0 \right\}.$$

We remark that $\mathcal{W}^m(\mathcal{T}) = \mathcal{W}^m(\mathcal{T},\emptyset)$. It is easily seen that the exterior derivative, applied to each cell, defines a linear mapping $\mathsf{d}^m : \mathcal{W}^m(\mathcal{T},\mathcal{U}) \to \mathcal{W}^{m+1}(\mathcal{T},\mathcal{U})$. Hence we introduce the differential complex of Whitney forms:

$$0 \leftarrow \mathcal{W}^n(\mathcal{T},\mathcal{U}) \xleftarrow{\mathsf{d}^{n-1}} \dots \xleftarrow{\mathsf{d}^0} \mathcal{W}^0(\mathcal{T},\mathcal{U}) \leftarrow 0. \tag{48}$$

Next, a Hilbert space structure over $\mathcal{W}^m(\mathcal{T},\mathcal{U})$ is induced by the scalar product

$$\langle \phi, \psi \rangle_{L^2\Lambda^m(\mathcal{T})} := \sum_{C\in\mathcal{T}^n} \langle \phi_C, \psi_C \rangle_{L^2\Lambda^m(C)}, \quad \phi, \psi \in \mathcal{W}^m(\mathcal{T},\mathcal{U}).$$

This scalar product has been introduced earlier in Section 2. By $\mathcal{W}^m(\mathcal{T},\mathcal{U})_{L^2\Lambda^m}$ we denote the Hilbert space that results from equipping $\mathcal{W}^m(\mathcal{T},\mathcal{U})$ with that scalar product. We have a Hilbert complex

$$0 \leftarrow \mathcal{W}^n(\mathcal{T},\mathcal{U})_{L^2\Lambda^n} \xleftarrow{\mathsf{d}^{n-1}} \dots \xleftarrow{\mathsf{d}^0} \mathcal{W}^0(\mathcal{T},\mathcal{U})_{L^2\Lambda^0} \leftarrow 0. \tag{49}$$

We let $\mu^{\mathcal{W}}_{\mathcal{T},\mathcal{U}} > 0$ denote the Poincaré-Friedrichs constant of this Hilbert complex. Hence, for every $\phi \in \mathsf{d}^m \mathcal{W}^m(\mathcal{T},\mathcal{U})$ there exists $\Phi \in \mathcal{W}^m(\mathcal{T},\mathcal{U})$ such that

$$\mathsf{d}^m \Phi = \phi, \quad \|\Phi\|_{L^2\Lambda^m} \leq \mu^{\mathcal{W}}_{\mathcal{T},\mathcal{U}} \|\phi\|_{L^2\Lambda^{m+1}}.$$

Recall the differential complex

$$0 \to \Gamma^0(\mathcal{T}^n) \xrightarrow{\mathsf{T}^n_0} \dots \xrightarrow{\mathsf{T}^1_0} \Gamma^0(\mathcal{T}^0) \to 0. \tag{50}$$

We equip each space $\Gamma^0(\mathcal{T}^m)$ with the scalar product

$$\langle \omega, \eta \rangle_{-h} := \sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} h_C^{m-n} \omega_C \eta_C, \quad \omega, \eta \in \Gamma^0(\mathcal{T}^m).$$

This makes (50) into a Hilbert complex and we write $\mu^{\Gamma}_{\mathcal{T},\mathcal{U}} > 0$ for its Poincaré-Friedrichs constant. We relate $\mu^{\Gamma}_{\mathcal{T},\mathcal{U}}$ to the Poincaré-Friedrichs constant of (49). For the proof of the following theorem, we refer to [20].

Theorem 5.1. *We have $\mu^{\Gamma}_{\mathcal{T},\mathcal{U}} \leq \mu \left(\mu^{m-1}_{\mathcal{W}}\right)^2$, where μ depends only on $\mu_{\mathcal{T}}$. In particular, for $1 \leq m \leq n$ there exists a linear operator $\mathscr{E}^m : \Gamma^0(\mathcal{T}^{m-1}) \to \Gamma^0(\mathcal{T}^m)$ such that for all $\xi \in \Gamma^0(\mathcal{T}^m)$ we have $\mathsf{T}^m_k \mathscr{E}^m \mathsf{T}^m_k \xi = \mathsf{T}^m_k \xi$, and*

$$\sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} h_C^{m-n} \|(\mathscr{E}^m \xi)_C\|^2_{L^2\Lambda^0(C)} \leq \left(\mu^{\Gamma}_{\mathcal{T},\mathcal{U}}\right)^2 \sum_{F \in \mathcal{T}^{m-1} \setminus \mathcal{U}^{m-1}} h_F^{m-n-1} \|\xi_F\|^2_{L^2\Lambda^0(F)}.$$

Remark 5.2. *We express the constants of our generalized Poincaré-Friedrichs inequalities in terms of $\mu^{\mathcal{W}}_{\mathcal{T},\mathcal{U}}$, but we do not attempt to determine $\mu^{\mathcal{W}}_{\mathcal{T},\mathcal{U}}$ any further here. In typical applications, $\mu^{\mathcal{W}}_{\mathcal{T},\mathcal{U}}$ depends on the mesh quality and the Poincaré-Friedrichs constant μ_{Ω,Γ_N} of the L^2 de Rham complex.*

6. Derivation of Poincaré-Friedrichs-type inequalities

In this section we reduce the Poincaré-Friedrichs constants in Hilbert complexes of discrete distributional differential forms to the Poincaré-Friedrichs constant $\mu^{\mathcal{W}}_{\mathcal{T},\mathcal{U}}$. The inequalities are proven with respect to the mesh-dependent scalar product

$$\langle \omega, \eta \rangle_{-h} := \sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} h_C^{m-n} \langle \omega_C, \eta_C \rangle_{L^2\Lambda^k(C)}, \quad \omega, \eta \in \Lambda^k_{-1}(\mathcal{T}^m). \tag{51}$$

This induces scalar products

$$\langle \omega, \eta \rangle_{-h} := \sum_{i=0}^{b-1} \langle \omega^{n-m+i}, \eta^{n-m+i} \rangle_{-h}, \quad \omega, \eta \in \Lambda^k_{-b}(\mathcal{T}^m),$$
$$\langle \omega, \eta \rangle_{-h} := \sum_{i=0}^{b-1} \langle \omega^{m+i}, \eta^{m+i} \rangle_{-h}, \quad \omega, \eta \in \Gamma^k_{-b}(\mathcal{T}^m),$$

where we use the definition of $\Lambda^k_{-b}(\mathcal{T}^m)$ and $\Gamma^k_{-b}(\mathcal{T}^m)$ as direct sums.

Example 6.1. *We revisit a motivational example. Suppose that $\mathcal{T}$ triangulates a simply connected domain and that $\mathcal{U} = \emptyset$. Let $\omega \in \mathsf{d}^0\Lambda^0(\mathcal{T}^n)$ be the gradient of a function in the conforming finite element space $\Lambda^0(\mathcal{T}^n)$. We show how to construct a preimage under the exterior derivative. We let $\xi^0 = \mathsf{P}^n_0 \omega$. Then $\mathsf{D}^n_0 \xi^0 = \omega$*

because $\mathsf{D}^n_1\omega = 0$ *and finite element spaces are an exact sequence on each simplex. However,* $\xi^0 \in \Lambda^0_{-1}(\mathcal{T}^n)$ *is discontinuous in general. We write* $\eta = \omega - \mathsf{d}^0\xi^0$*. Then*

$$\eta = \omega - \mathsf{D}^n_0\mathsf{P}^n_0\omega + \mathsf{T}^n_0\mathsf{P}^n_0\omega = \mathsf{T}^n_0\xi^0, \quad \mathsf{D}^{n-1}_0\mathsf{T}^n_0\xi^0 = \mathsf{T}^n_1\mathsf{D}^n_0\xi^0 = \mathsf{T}^n_1\omega = 0.$$

Hence $\eta \in \Gamma^0(\mathcal{T}^{n-1})$ *with* $\mathsf{T}^{n-1}_0\eta = \mathsf{T}^{n-1}_0\mathsf{T}^n_0\xi^0 = 0$*. Note that* η *represents the interelement jumps of* ξ^0*, which are piecewise constant due to* $\mathsf{D}^{n-1}_0\eta = 0$*, and which can hence be identified with an* $n-1$ *chain of the triangulation. But since we assume the domain to be simply connected,* $\mathsf{T}^{n-1}_0\eta = 0$ *already guarantees that there exists* $\widetilde{\xi} \in \Gamma^0(\mathcal{T}^n)$ *such that* $\mathsf{T}^n_0\widetilde{\xi} = \eta$*. But then*

$$\mathsf{d}^0\left(\xi^0 + \widetilde{\xi}\right) = \omega - \eta + \mathsf{d}^0\widetilde{\xi} = \omega - \eta + \mathsf{T}^n_0\widetilde{\xi} = \omega.$$

We set $\xi := \xi^0 + \widetilde{\xi}$*, so* $\mathsf{d}^0\xi = \omega$*. This is the desired preimage in the conforming finite element space. Note that the only non-local operation was finding* $\widetilde{\xi}$*, which is independent of any polynomial degree. In the remainder of this section, we extend this simple example to general discrete distributional differential forms.*

Throughout this section we assume that $\omega \in \mathsf{d}^k\Lambda^k_{-k-1}(\mathcal{T}^n)$ is a discrete distributional differential form within the range of the exterior derivative $\mathsf{d}^k : \Lambda^k_{-k-1}(\mathcal{T}^n) \to \Lambda^{k+1}_{-k-2}(\mathcal{T}^n)$. There is a unique way to write ω as

$$\omega = \omega_0 + \cdots + \omega_{k+1}, \quad \omega_i \in \Lambda^{k+1-i}_{-1}(\mathcal{T}^{n-i}), \quad 0 \le i \le k+1.$$

We construct $\xi \in \Lambda^k_{-b}(\mathcal{T}^n)$ such that $\mathsf{d}^k\xi = \omega$ and keep track of inequalities; this will prove the generalized Poincaré-Friedrichs inequalities.

Explicitly, we define $\xi \in \Lambda^k_{-k-1}(\mathcal{T}^n)$ as $\xi := \xi^0 + \cdots + \xi^k + \widetilde{\xi}$, where we first set $\xi^0 := \mathsf{P}^n_k\omega^0 \in \Lambda^k_{-1}(\mathcal{T}^n)$, then recursively define

$$\xi^i := (-1)^i\mathsf{P}^{n-i}_{k-i}\omega^i + \mathsf{P}^{n-i}_{k-i}\mathsf{T}^{n-i+1}_{k-i+1}\xi^{i-1} \in \Lambda^{k-i}_{-1}(\mathcal{T}^{n-i}), \tag{52}$$

for $1 \le i \le k$, and eventually fix

$$\widetilde{\xi} := (-1)^{k+1}\mathscr{E}^m\omega^{k+1} - \mathscr{E}^m\mathsf{T}^{n-k}_0\xi^k \in \Gamma^0(\mathcal{T}^{n-k}). \tag{53}$$

Here we have used the horizontal antiderivative P^m_k and the operator $\mathscr{E}^m$ from Theorem 5.1. The basic idea has been standard in differential topology [4, II.9] for a long time. First we verify the following identity.

Lemma 6.2. *Let* ω *and* ξ *be defined as above. Then* $\mathsf{d}^k\xi = \omega$*.*

Proof. First, we have $\mathsf{D}^n_{k+1}\omega_0 = 0$ by assumption. The local exactness condition thus implies $\mathsf{D}^n_k\xi^0 = \omega^0$. We moreover have, since $\mathsf{d}^k\omega = 0$, that

$$\begin{aligned}-\mathsf{D}^{n-1}_k\left(\omega^1 + \mathsf{T}^n_k\xi^0\right) &= -\mathsf{D}^{n-1}_k\omega^1 - \mathsf{D}^{n-1}_k\mathsf{T}^n_k\xi^0 \\ &= -\mathsf{D}^{n-1}_k\omega^1 - \mathsf{T}^n_{k+1}\mathsf{D}^n_k\xi^0 = -\mathsf{D}^{n-1}_k\omega^1 - \mathsf{T}^n_{k+1}\omega^0 = 0.\end{aligned}$$

Next we use an induction argument. Let us assume that we already have

$$\begin{aligned}\omega - \mathsf{d}^k\left(\xi^0 + \cdots + \xi^i\right) &= (-1)^i\mathsf{T}^{n-i}_{k-i}\xi^i + \omega^{i+1} + \cdots + \omega^b, \\ \mathsf{D}^{n-i-1}_{k-i-1}\omega^{i+1} &= (-1)^{i+1}\mathsf{D}^{n-i-1}_{k-i-1}\mathsf{T}^{n-i}_{k-i}\xi^i,\end{aligned}$$

for some $0 \le i < k$. But now

$$\mathsf{D}^{n-i-1}_{k-i-1}\xi^{i+1} = (-1)^{i+1}\omega^{i+1} - \mathsf{T}^{n-i}_{k-i}\xi^i.$$

Thus we find

$$\omega - \mathsf{d}^k \left(\xi^0 + \dots + \xi^{i+1} \right) = (-1)^{i+1} \mathsf{T}^{n-i-1}_{k-i-1} \xi^{i+1} + \omega^{i+2} + \dots + \omega^b \in \mathsf{d}\Lambda^k_{-b}(\mathcal{T}^n),$$

and

$$\begin{aligned}\mathsf{D}^{n-i+2}_{k-i+2} \left((-1)^{i+1} \mathsf{T}^{n-i-1}_{k-i-1} \xi^{i+1} + \omega^{i+2} \right) &= (-1)^{i+1} \mathsf{D}^{n-i-2}_{k-i-1} \mathsf{T}^{n-i-1}_{k-i-1} \xi^{i+1} + \mathsf{D}^{n-i-2}_{k-i-1} \omega^{i+2} \\ &= \mathsf{T}^{n-i+1}_{k-i} \omega^{i+1} + \mathsf{D}^{n-i-2}_{k-i-1} \omega^{i+2} = 0.\end{aligned}$$

Hence the assumptions for index i hold again for the index $i+1 \leq k$.

Iteration of this argument eventually provides

$$\eta := \omega - \mathsf{d}^k \left(\xi^0 + \dots + \xi^k \right) = \omega^{k+1} + (-1)^{k+1} \mathsf{T}^{n-k}_0 \xi^k,$$

which we study in further detail. We see that $\mathsf{D}^{n-k-1}_0 \eta = 0$, and $\mathsf{T}^{n-k-1}_0 \eta = 0$. In particular $\eta \in \Gamma^0(\mathcal{T}^{n-k-1})$. Since $\eta \in \mathsf{d}^k \Lambda^k_{-k-1}(\mathcal{T}^n)$ by construction, we know that η is orthogonal to $\mathfrak{H}^{k+1}_{-k-2}(\mathcal{T}^n) = \mathfrak{C}^{k+1}_{-k-2}(\mathcal{T}^{n-k-1})$. We need to show that η is orthogonal to $\mathfrak{C}^0(\mathcal{T}^{n-k-1})$. To see this, let $p \in \mathfrak{C}^{k+1}_{-k-2}(\mathcal{T}^{n-k-1})$ be fixed. Then there is a unique way to write $p = p^0 + \dots + p^k + p^{k+1}$ with $p^i \in \Lambda^{k+i}(\mathcal{T}^{n+i})$ for each $0 \leq i \leq k+1$. From Lemma 2.9 we conclude that $p^0 \in \mathfrak{C}^0(\mathcal{T}^{n-k})$. Hence

$$\begin{aligned}\left\langle p^0, (-1)^{k+1} \mathsf{T}^{n-k}_0 \xi^k + \omega^{k+1} \right\rangle_{-h} &= \left\langle p, (-1)^{k+1} \mathsf{T}^{n-k}_0 \xi^k + \omega^{k+1} \right\rangle_{-h} \\ &= \left\langle p, \omega - \mathsf{d}^k \left(\xi^0 + \dots + \xi^k \right) \right\rangle_{-h} = \langle p, \omega \rangle_{-h} = 0\end{aligned}$$

by assumption on ω. Thus η is orthogonal to $\mathfrak{C}^0(\mathcal{T}^{n-k-1})$, and we conclude further that $(-1)^{k+1} \mathsf{T}^{n-k}_0 \left(\widetilde{\xi} + \xi_k \right) = \omega^{k+1}$. This completes the proof. $\square$

Next we show that the preimage ξ is uniformly bounded in terms of ω with respect to the $\| \cdot \|_{-h}$ norm. In fact, we show a slighly more general result, parameterized by a real number $\alpha \in \mathbb{R}$, which will be of technical interest in the sequel.

Lemma 6.3. *Let ω and ξ be as above. For any $\alpha \in \mathbb{R}$ there exists μ_α, depending only on $\hat{\mu}$, $\grave{\mu}$, and α, such that*

$$\sum_{C \in \mathcal{T}^{n-i}} h_C^{\alpha-i} \left\| \xi^i_C \right\|^2_{L^2\Lambda^{k-i}(C)} \leq \mu_\alpha \sum_{j=0}^{i} \sum_{C \in \mathcal{T}^{n-j}} h_C^{\alpha-j+2} \left\| \omega^j_C \right\|^2_{L^2\Lambda^{k-j+1}(C)}. \tag{54}$$

Moreover, with a constant $\widetilde{\mu}$ that depends only on μ_0, μ_1, $\grave{\mu}$, and $\mu^\Gamma_{\mathcal{T},\mathcal{U}}$, we have

$$\sum_{C \in \mathcal{T}^{n-k}} h_C^{-k} \left\| \widetilde{\xi}_C \right\|^2_{L^2\Lambda^0(C)} \leq \widetilde{\mu} \sum_{i=0}^{k+1} \sum_{C \in \mathcal{T}^{n-k-1}} h_C^{-i} \left\| \omega^i_C \right\|^2_{L^2\Lambda^{k-i+1}(C)} \tag{55}$$

Proof. From Theorem 4.1 we find

$$\sum_{C \in \mathcal{T}^n} h_C^{\alpha} \left\| \xi^0_C \right\|^2_{L^2\Lambda^k(C)} \leq \hat{\mu} \sum_{C \in \mathcal{T}^n} h_C^{\alpha+2} \left\| \omega^0_C \right\|^2_{L^2\Lambda^{k+1}(C)}. \tag{56}$$

This shows (54) in the case $i = 0$. We proceed with an induction argument. Suppose that (54) holds for some index $0 \leq i \leq k-1$. By construction of ξ^{i+1} we then have

$$\begin{aligned}
&\sum_{C \in \mathcal{T}^{n-i-1}} h_C^{\alpha-i-1} \left\| \xi_C^{i+1} \right\|^2_{L^2\Lambda^{k-i-1}(C)} \\
&= \sum_{C \in \mathcal{T}^{n-i-1}} h_C^{\alpha-i-1} \left\| \mathsf{P}^{n-i-1}_{k-i-1} \omega_C^{i+1} - \mathsf{P}^{n-i-1}_{k-i-1} (\mathsf{T}^{n-i}_{n-i} \xi^i)_C \right\|^2_{L^2\Lambda^{k-i-1}(C)} \\
&\leq \hat\mu \sum_{C \in \mathcal{T}^{n-i-1}} h_C^{\alpha-i+1} \left\| \omega_C^{i+1} - (\mathsf{T}^{n-i}_{n-i} \xi^i)_C \right\|^2_{L^2\Lambda^{k-i}(C)} \\
&\leq 2\hat\mu \sum_{C \in \mathcal{T}^{n-i-1}} h_C^{\alpha-i+1} \left\| \omega_C^{i+1} \right\|^2_{L^2\Lambda^{k-i}(C)} + 2\hat\mu\grave\mu \sum_{C \in \mathcal{T}^{n-i}} h_C^{\alpha-i} \left\| \xi_C^i \right\|^2_{L^2\Lambda^{k-i}(C)} \\
&\leq \mu_\alpha \sum_{j=0}^{i+1} \sum_{C \in \mathcal{T}^{n-j}} h_C^{\alpha-j+2} \left\| \omega_C^j \right\|^2_{L^2\Lambda^{k-j+1}(C)},
\end{aligned}$$

where μ_α is a constant as in the statement of the theorem. Here we have used that (54) holds for i, and we conclude that (54) holds for $i+1$ too. Iterative application of this observation shows (54) for all $0 \leq i \leq k$.

In order to show (55), let $\eta \in \Gamma^0(\mathcal{T}^{n-k-1})$ be as in the proof of the Theorem 6.2. Using Theorem 5.1 we first observe

$$\sum_{C \in \mathcal{T}^{n-k}} h_C^{-k} \left\| \widetilde\xi_C \right\|^2_{L^2\Lambda^0(C)} \leq \left(\mu^\Gamma_{\mathcal{T},\mathcal{U}}\right)^2 \sum_{C \in \mathcal{T}^{n-k-1}} h_C^{-k-1} \left\| \eta_C \right\|^2_{L^2\Lambda^0(C)}.$$

From definitions we get

$$\sum_{C \in \mathcal{T}^{n-k-1}} h_C^{-k-1} \left\| \eta_C \right\|^2_{L^2\Lambda^0(C)} = \sum_{C \in \mathcal{T}^{n-k-1}} h_C^{-k-1} \left\| \omega_C^{k+1} - (\mathsf{T}^{n-k}_0 \xi^k)_C \right\|^2_{L^2\Lambda^0(C)}.$$

We now see

$$\sum_{C \in \mathcal{T}^{n-k-1}} h_C^{-k-1} \left\| (\mathsf{T}^{n-k}_0 \xi^k)_C \right\|^2_{L^2\Lambda^0(C)} \leq \grave\mu \sum_{C \in \mathcal{T}^{n-k}} h_C^{-k-2} \left\| \xi_C^k \right\|^2_{L^2\Lambda^0(C)}.$$

We apply (54) with $i = k$ and $\alpha = -2$ to obtain (55). This completes the proof. □

Corollary 6.4. *The Hilbert complex*

$$\Lambda^0_{-1}(\mathcal{T}^n) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \Lambda^n_{-n-1}(\mathcal{T}^n)$$

satisfies Poincaré-Friedrichs inequalities with respect to the scalar product $\langle\cdot,\cdot\rangle_{-h}$. The Poincaré-Friedrichs constant depends only on $\mu^\Gamma_{\mathcal{T},\mathcal{U}}$, $\mu_\mathcal{T}$, R.

We have proven Poincaré-Friedrichs-type inequalities for the "maximal" complexes of discrete distributional differential forms (33) / (34). A modification of this construction allows us to derive analogous inequalities for the complexes (29) – (32). The guideline will be the following: if certain components of ω are zero, can we construct ξ with the appropiate components zero? A Poincaré-Friedrichs-type inequality follows again by carefully keeping track of the inequalities.

For example, let us suppose that $\omega \in \Lambda^{k+1}_{-k+n-m-1}(\mathcal{T}^m)$, which means that $\omega_i = 0$ for $0 \leq i \leq n-m$. It is easily seen that $\xi_i = 0$ for $0 \leq i \leq n-m$ in that case. In other words, $\omega \in \Lambda^{k+1}_{-k+n-m+1}(\mathcal{T}^m)$ implies $\xi \in \Lambda^{k+1}_{-k+n-m+1}(\mathcal{T}^m)$.

Analogously, if $\omega \in \Lambda^{k+1}_{-b}(\mathcal{T}^n)$ for some $2 \leq b \leq k+2$, then there should exist a preimage under the distributional exterior derivative in $\Lambda^k_{-b+1}(\mathcal{T}^n)$ that satisfies a generalized Poincaré-Friedrichs inequality. We construct such a preimage from ξ.

If $\omega \in \Lambda^{k+1}_{-b}(\mathcal{T}^n)$, then $\omega = \omega^0 + \cdots + \omega^{-b+1}$, and we have

$$\xi^i = \mathsf{P}^{n-i}_{k-i} \mathsf{T}^{n-i+1}_{k-i+1} \xi^{i-1}, \quad b \leq i \leq k,$$
$$\widetilde{\xi} = -\mathscr{P}^{n-k} \mathsf{T}^{n-k}_{0} \xi^{k}.$$

Unfolding this recursive relation, we find that

$$\xi^i = \mathsf{P}^{n-i}_{k-i} \mathsf{T}^{n-i+1}_{k-i+1} \cdots \mathsf{P}^{n-b}_{k-b} \mathsf{T}^{n-b+1}_{k-b+1} \xi^{b-1}, \quad b \leq i \leq k,$$
$$\widetilde{\xi} = -\mathscr{P}^{n-k} \mathsf{T}^{n-k}_{0} \mathsf{P}^{n-k}_{0} \mathsf{T}^{n-k+1}_{1} \cdots \mathsf{P}^{n-b}_{k-b} \mathsf{T}^{n-b+1}_{k-b+1} \xi^{b-1}.$$

By iterative application of Lemma 2.6 we find

$$R_{k,i} \ldots R_{k,k+1} \xi \in \Lambda^k_{-i}(\mathcal{T}^n), \quad \mathsf{d}^k R_{k,i} \cdots R_{k,k+1} \xi = \omega$$

for $b \leq i \leq k+1$. In particular, the case $i = b$ provides a member of $\Lambda^k_{-b+1}(\mathcal{T}^n)$ whose distributional exterior derivative equals ω. Going further, we have

$$R_{k,b} \ldots R_{k,k+1} \xi = \xi^0 + \cdots + \xi^{b-2} + \theta^{b-1} + \cdots + \theta^k + \widetilde{\theta},$$

where we write

$$\theta^i = (-1)^{i-b} \mathsf{D}^{n-b+2}_{k-b+1} \mathsf{E}^{n-b+2}_{k-b+1} \cdots \mathsf{D}^{n-i+1}_{k-i} \mathsf{E}^{n-i+1}_{k-i} \xi^i, \quad b-1 \leq i \leq k,$$
$$\widetilde{\theta} = (-1)^{k-b+1} \mathsf{D}^{n-b+2}_{k-b+1} \mathsf{E}^{n-b+2}_{k-b+1} \cdots \mathsf{D}^{n-k+1}_{0} \mathsf{E}^{n-k+1}_{0} \widetilde{\xi}^{i}.$$

For the terms $\xi^0, \ldots, \xi^{b-2}$ we can still use (54), but we need to derive new estimates for the terms $\theta^{b-1}, \ldots, \theta^k$. We will see that they satisfy very similar estimates.

We derive estimates via Lemma 4.1. When $\alpha \in \mathbb{R}$ we obtain that

$$\sum_{C \in \mathcal{T}^{n-b+2}} h_C^{\alpha-b+2} \left\| \theta^i_C \right\|^2_{L^2\Lambda^{k-b+2}(C)} \leq \sum_{C \in \mathcal{T}^{n-i}} h_C^{\alpha-i} \left\| \xi^i_C \right\|^2_{L^2\Lambda^{k-i}(C)}$$
$$\leq \sum_{C \in \mathcal{T}^{n-b+1}} h_C^{\alpha-b+1} \left\| \xi^{b-1}_C \right\|^2_{L^2\Lambda^{k-b-1}(C)} \leq \sum_{j=0}^{b-1} \sum_{C \in \mathcal{T}^{n-j}} h_C^{\alpha-b+3} \left\| \omega^j_C \right\|^2_{L^2\Lambda^{k-j}(C)}.$$

Next, with application of Lemma 6.3, and Lemma 5.1, we find

$$\sum_{C \in \mathcal{T}^{n-b+2}} h_C^{-b+2} \left\| \widetilde{\theta}_C \right\|^2_{L^2\Lambda^{k-b+2}(C)} \leq \sum_{C \in \mathcal{T}^{n-k}} h_C^{-k} \left\| \widetilde{\xi}_C \right\|^2_{L^2\Lambda^{0}(C)}$$
$$\leq \sum_{C \in \mathcal{T}^{n-k}} h_C^{-k-2} \left\| \xi^k_C \right\|^2_{L^2\Lambda^{0}(C)} \quad \leq \sum_{C \in \mathcal{T}^{n-b-1}} h_C^{-b-1} \left\| \xi^{b-1}_C \right\|^2_{L^2\Lambda^{k-b-1}(C)}$$
$$\leq \sum_{j=0}^{b-1} \sum_{C \in \mathcal{T}^{n-j}} h_C^{-b+1} \left\| \omega^j_C \right\|^2_{L^2\Lambda^{k-j}(C)}.$$

We have thus constructed an element of $\Lambda^k_{-b+1}(\mathcal{T}^n)$ whose distributional exterior derivative equals ω, and have derived the desired inequalities.

Finally, suppose that $\omega^1 = 0$, so that $\omega \in \Lambda^{k+1}_{-1}(\mathcal{T}^n)$. Since $\omega \in \mathsf{d}^k \Lambda^k_{-1}(\mathcal{T}^n)$, we find that $\omega \in \Lambda^{k+1}(\mathcal{T}^n)$ and $\omega \in \mathsf{d}^k \Lambda^k(\mathcal{T}^n)$. The construction of ξ then assures that $\xi \in \Lambda^k(\mathcal{T}^n)$. Conclusively, we have derived bounds for the Poincaré-Friedrichs constant of finite element de Rham complexes.

We have thus proven the following main result.

Theorem 6.5. *The Hilbert complexes*

$$\dots \xrightarrow{\mathsf{d}^{k+n-m-2}} \Lambda^{k-1}(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{k+n-m-1}} \Lambda^{k}_{-1}(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{k+n-m}} \Lambda^{k+1}_{-2}(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{k+n-m+1}} \dots$$

and

$$\dots \xrightarrow{\mathsf{d}^{k+n-m-2}} \Gamma^{k-1}(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{k+n-m-1}} \Gamma^{k}_{-1}(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{k+n-m}} \Gamma^{k+1}_{-2}(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{k+n-m+1}} \dots$$

satisfy Poincaré-Friedrichs inequalities with respect to the scalar product $\langle\cdot,\cdot\rangle_{-h}$, where the constants only depend on $\mu^{\Gamma}_{\mathcal{T},\mathcal{U}}$, $\mu_{\mathcal{T}}$, and R.

Remark 6.6. *Braess and Schöberl use a scaling argument to prove Poincaré-Friedrichs inequalities with respect to $\langle\cdot,\cdot\rangle_h$ (see* (16)*) in* [5, Section 3.4] *when the underlying triangulation is a local patch. An easy scaling argument also shows that their Lemma 9 holds similarly with our scalar product $\langle\cdot,\cdot\rangle_{-h}$ for the distributional finite element complex over a local patch. In the light of this, we are inclined to consider $\langle\cdot,\cdot\rangle_{-h}$ as the "natural" scalar product for distributional finite element spaces.*

Example 6.7. *Let $\mathcal{T}$ triangulate a simply connected domain and let $\mathcal{U} = \emptyset$. The section shows how to construct $\sigma \in \Lambda^{n-1}(\mathcal{T}^n)$ for given $f \in \Lambda^n(\mathcal{T}^n)$ such that $\mathsf{d}^{n-1}\sigma = f$. The only global operation is finding, for a 0-chain $s \in \mathcal{C}_0(\mathcal{T})$ (i.e. a linear combination of points), a 1-chain $s \in \mathcal{C}_1(\mathcal{T})$ (i.e. a linear combination of edges) such that $\partial S = s$. The condition number of the latter global problem with respect $\|\cdot\|_{-h}$ depends only on the shape constant $\mu_{\mathcal{T}}$, but not on any polynomial degree.*

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