HIGHER ORDER FINITE ELEMENT DE RHAM COMPLEXES, PARTIALLY LOCALIZED FLUX RECONSTRUCTIONS, AND APPLICATIONS

MARTIN WERNER LICHT

Abstract. We construct finite element de Rham complexes of higher and possibly non-uniform polynomial order in finite element exterior calculus. Starting from the finite element differential complex of lowest-order, namely the complex of Whitney forms, we incrementally construct the higher order complexes by adjoining exact local complexes associated to simplices. A commuting canonical interpolant is defined. On the one hand, this research provides a base for studying *hp*-adaptive methods in FEEC. On the other hand, our construction of higher order spaces enables a new tool in numerical analysis which we call "partially localized flux reconstruction". One major application of this concept is equilibrated a posteriori error estimators. In particular, we generalize the Braess-Schöberl error estimator to edge elements of higher and possibly non-uniform order.

1. Introduction

The formalism of differential complexes offers a theoretical approach to many partial differential equations in physics and engineering. Maxwell's equations in electromagnetism are perhaps the most prominent example. Numerical analysis has embraced differential complexes of finite element spaces in the design of mixed finite element methods in computational electromagnetism (28; 34; 4). Whereas many contributions in analysis utilize classical vector calculus, the calculus of differential forms in differential geometry enables a unified treatment of differential operators such as the gradient, the curl, or the divergence. The wide adoption of the calculus of differential forms in the theory of partial differential equations motivates the study of finite element differential forms in numerical analysis (10; 28). This line of thought has culminated in *finite element exterior calculus* (FEEC, (4; 6)), which is the mathematical formalism that we adopt in this contribution.

At the discrete level, finite element exterior calculus considers piecewise polynomial differential forms. Research efforts have focused on spaces of uniform polynomial order (4; 5), but have given considerably less attention to spaces with spatially varying polynomial order (but see 27). Finite element spaces of the latter kind, however, are constitutive for p-adaptive and hp-adaptive finite element methods (38; 22; 35). We recall that h-adaptive methods refine the mesh locally but keep the polynomial order fixed, that p-adaptive methods keep the mesh fixed but locally

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increase the polynomial order, and that *hp*-adaptive methods combine local mesh refinement and variation of the polynomial order. The latter form of adaptivity allows for efficient approximation of functions with spatially varying smoothness or isolated singularities, for example by Lagrange elements with non-uniform polynomial order. The theory of *hp*-adaptive mixed finite element methods in numerical electromagnetism utilizes differential complexes of spaces of non-uniform polynomial order, which include generalizations of Nédélec elements and Raviart-Thomas elements (33; 1; 21; 37). The major part of these research efforts has been formalized in terms of classical vector calculus.

In this paper we study the algebraic and structural properties of finite element de Rham complexes of higher polynomial order. We develop a formalism for finite element spaces of non-uniform polynomial order and construct a commuting interpolant. This prepares future research on *hp*-adaptive methods in FEEC. The main result of our research effort though are algorithms for *partially localized flux reconstruction*. Such algorithms are crucial for the efficient implementation of equilibrated a posteriori error estimators (13). One preliminary achievement in this regard is generalizing the locally constructed Braess-Schöberl error estimator for edge elements (14) to the higher order case.

It is common practice in literature on *hp*-FEM to characterize approximation spaces by assigning a polynomial order $r_S \in \mathbb{N}_0$ to each simplex S of the mesh, such that simplices have an associated polynomial order at least as large as the ones associated to their subsimplices. This specifies spaces of functions whose trace on each simplex S is a polynomial of order at most r_S. For finite element exterior calculus we extend this concept: on each simplex we fix not only the polynomial order but also the choice between the $\mathcal{P}_r$-family and $\mathcal{P}_r^-$-family of finite element spaces. For example, this allows to choose between Raviart-Thomas spaces and Brezzi-Douglas-Marini spaces being associated to a triangle.

We build on the intuition that finite element de Rham complexes of higher (uniform or non-uniform) polynomial order are constructed from the lowest-order finite element de Rham complex by local augmentations with local higher-order complexes. A variant of this idea was already used by (37) and (41). We may recall that the lowest-order finite element de Rham complex is precisely the differential complex of Whitney forms (10). In three dimensions, the latter translates into the well-known differential complex

$$\mathcal{P}_1(\mathcal{T}) \xrightarrow{\text{grad}} \mathbf{Nd}_0(\mathcal{T}) \xrightarrow{\text{curl}} \mathbf{RT}_0(\mathcal{T}) \xrightarrow{\text{div}} \mathcal{P}_{0,DC}(\mathcal{T}) \tag{1}$$

with respect to a triangulation $\mathcal{T}$ of a three-dimensional domain (10; 4). Here, $\mathcal{P}_1(\mathcal{T})$ denotes piecewise affine Lagrange elements, $\mathbf{Nd}_0(\mathcal{T})$ denotes lowest-order Nédélec elements, $\mathbf{RT}_0(\mathcal{T})$ denotes lowest-order Raviart-Thomas elements, and $\mathcal{P}_{0,DC}(\mathcal{T})$ is spanned by the piecewise constant functions.

For a simple example of how to augment this complex with a local higher order sequence, we fix $r \in \mathbb{N}$ and a tetrahedron $T \in \mathcal{T}$ and consider the differential complex

$$\mathring{\mathcal{P}}_{r+1}(T) \xrightarrow{\text{grad}} \mathring{\mathbf{Nd}}_r(T) \xrightarrow{\text{curl}} \mathring{\mathbf{RT}}_r(\mathcal{T}) \xrightarrow{\text{div}} \mathcal{P}_r(T) \cap \ker \textstyle\int_T . \tag{2}$$

The first three spaces are the higher order Lagrange space, the Nédélec space, and the Raviart-Thomas space over T with Dirichlet, tangential, and normal boundary conditions, respectively, along ∂T. The final space is the order r polynomials over T with vanishing mean value. This sequence is exact and supported only over T, and we can augment the lowest-order complex (1) by taking the direct sum with (2).

Similarly we may associate an exact finite element sequence to any lower dimensional subsimplex and extend the spaces onto the local neighborhood; when we extend the spaces to spaces on the local patch then the latter do generally not compose a differential complex unless the extension operators commute with the exterior derivative, but if the choice of finite element spaces reflects the inclusion ordering of simplices, then the global finite element spaces constitute a differential complex.

The degrees of freedom of the global higher order spaces are the direct sum of the degrees of freedom for the Whitney forms and the degrees of freedom of the local higher order spaces. We define the commuting canonical interpolant onto the finite element de Rham complex following the strategy of (23), which utilizes the Hodge decomposition of the degrees of freedom. Our canonical interpolant satisfies all the significant properties of their counterpart for spaces of uniform type (4); in particular, it may serve as a component in the smoothed projection (20).

Apart from relating FEEC and *hp*-adaptive FEM, our framework enables a new tool in finite element methods. Our construction of finite element spaces by augmenting the lowest-order finite element space gives a formalism to describe partially localized flux reconstructions. In this context, *flux reconstruction* refers to computing a generalized inverse of the exterior derivative between finite element spaces of differential forms. To formulate an example in the language of vector calculus, one might want to compute a generalized inverse for the mapping $\operatorname{curl} : \mathbf{Nd}_r(\mathcal{T}) \to \mathbf{RT}_r(\mathcal{T})$ from order r Nédélec elements to order r Raviart-Thomas elements.

Algorithmically we can tackle the problem either with a mixed finite element method or by solving normal equations. As such, both approaches will involve finite element spaces of higher order. Our framework, however, shows how to reduce the global problem to the lowest-order case. For example, assume that $\omega \in \mathbf{RT}_r(\mathcal{T})$ is the curl of a member of $\mathbf{Nd}_r(\mathcal{T})$. In this article we show how to decompose $\omega = \omega_0 + \operatorname{curl} \xi_r$, where $\omega_0 \in \mathbf{RT}_0(\mathcal{T})$ is the canonical interpolation of ω onto the lowest-order Raviart-Thomas space, and where $\xi_r \in \mathbf{Nd}_r(\mathcal{T})$ is constructed by solving independent local problems. These local problems are associated to single tetrahedra, and their size and well-posedness depends only on the local polynomial order and mesh quality; they are independent of each other and hence accessible to parallelization. It can be shown that there exists $\xi_0 \in \mathbf{Nd}_0(\mathcal{T})$ with $\operatorname{curl} \xi_0 = \omega_0$, and so $\xi := \xi_0 + \xi_r \in \mathbf{Nd}_r(\mathcal{T})$ satisfies $\operatorname{curl} \xi = \omega$. We compute the vector field ξ_0 by solving a global problem only on a smaller lowest-order space. The flux reconstruction is *partially localized* in the sense that only the lowest-order terms require a global computation.

A minor application of theoretical interest is determining the cohomology groups of finite element de Rham complex with varying polynomial order. Specifically, the canonical interpolant onto the Whitney forms induces isomorphisms on cohomology.

A major application, however, solves an open problem in the theory of a equilibrated posteriori error estimators. Those estimators have attracted persistent research efforts because they provide reliable and constant-free upper bounds for error of finite element methods (2; 36; 39). Efficient algorithms for finite element flux reconstruction are critical to make the estimator competitive in computations (12; 25; 7; 8). In the case of the Poisson problem, a fully localized flux reconstruction for the divergence operator $\operatorname{div} : \mathbf{RT}_r(\mathcal{T}) \to \mathcal{P}_{r-1,DC}(\mathcal{T})$ is possible if the Galerkin solution for the Poisson problem is given as additional information; the resulting estimator is competitive (12; 13; 16). Much less is known for equilibrated error estimators in numerical electromagnetism. Braess and Schöberl have introduced an equilibrated a posteriori error estimator for the curl-curl problem over lowest-order Nédélec elements (14). Analogously to the Poisson problem, they provide a fully localized flux reconstruction for the curl operator $\operatorname{curl} : \mathbf{Nd}_0(\mathcal{T}) \to \mathbf{RT}_0(\mathcal{T})$ which uses a Galerkin solution of the curl-curl-problem as additional information to achieve full localization. The generalization of this result to higher order edge elements has remained an open problem. But our partially localized flux reconstruction enables a *fully localized flux reconstruction* even in the higher order case. In effect we generalize equilibrated a posteriori error estimators for the curl-curl problem to the case of edge elements of higher and possibly non-uniform polynomial order.

The remainder of this article is structured as follows. In Section 2 we recapitulate smooth and polynomial differential forms. In Section 3 we consider exact sequences of polynomial differential forms over simplices. The complex of Whitney forms over a triangulation and the associated commuting interpolator are considered in Section 4. Finite element de Rham complexes of higher order are considered in Section 5. In Section 6 the partially localized flux reconstruction is introduced. Section 7 eventually demonstrates the application to the Braess-Schöberl error estimator. We finish with publication with some concluding remarks in Section 8.

2. Smooth and Polynomial Differential Forms

In this section we briefly recapitulate the calculus of differential forms on simplices. We subsequently give a summary of the $\mathcal{P}_r$ and $\mathcal{P}_r^-$ families of spaces of polynomial differential forms. As a general reference on differential forms we point out Agricola and Friedrich's monograph (26). Our discussion of polynomial differential forms is based on Arnold, Falk and Winther's seminal publication (4). We only give a small outline and refer the reader to these sources for a thorough treatment.

We agree on some notation. For $a \in \{0,1\}$, $k \in \mathbb{Z}$, and $n \in \mathbb{N}_0$ we let $\Sigma(a:k, 0:n)$ denote the set of strictly increasing mappings from $\{a,\dots,k\}$ to $\{0,\dots,n\}$. Note that $\Sigma(a:k,0:n) = \emptyset$ if $k > n$ and that $\Sigma(a:k,0:n) = \{\emptyset\}$ if $k < a$. For $n \in \mathbb{N}_0$ we let $A(n)$ denote the set of *multiindices* in $n+1$ variables, i.e. the set of functions from $\{0,\dots,n\}$ to $\mathbb{N}_0$. The absolute value of $\alpha \in A(n)$ is $|\alpha| := \alpha(0) + \cdots + \alpha(n)$. For $r \in \mathbb{N}$ we let $A(r,n)$ be the set of multiindices with absolute value r.

We let $T \subset \mathbb{R}^N$ be a fixed but arbitrary n-dimensional simplex. We henceforth write $\dim T$ for the dimension of any simplex, which one less than the number of its vertices. We introduce the n-dimensional volume $\operatorname{vol}(T)$ of T. We write $\Delta(T)$ for the set of subsimplices of T. If $F \in \Delta(T)$, then $\imath_{F,T} : F \to T$ denotes the inclusion in the sense of manifolds with corners. Note that for $F \in \Delta(T)$ and $f \in \Delta(F)$ we have $\imath_{f,T} = \imath_{F,T}\imath_{f,F}$. Throughout this article, we assume that each simplex is equipped with an arbitrary but fixed orientation.

We let $C^\infty(T)$ denote the space of smooth functions over T that are restrictions of a smooth function over $\mathbb{R}^n$. More generally, for $k \in \mathbb{Z}$ we let $C^\infty\Lambda^k(T)$ denote the space of smooth differential k-forms over T that are the pullback of a smooth differential k-form over $\mathbb{R}^n$ along the embedding of the simplex. We have $C^\infty\Lambda^0(T) = C^\infty(T)$ and $C^\infty\Lambda^k(T) = \emptyset$ for $k \notin \{0,\dots,n\}$. When $\omega \in C^\infty\Lambda^k(T)$ and $\eta \in C^\infty\Lambda^l(T)$, then $\omega \wedge \eta \in C^\infty\Lambda^{k+l}(T)$ denotes the exterior product. We recall that $\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega$. We also remember the exterior derivative

$$\mathsf{d}^k : C^\infty\Lambda^k(T) \to C^\infty\Lambda^{k+1}(T),$$

which satisfies the differential property $\mathsf{d}^{k+1}\mathsf{d}^k\omega = 0$ for all $\omega \in C^\infty\Lambda^k(T)$.

If $T' \subset \mathbb{R}^N$ is another simplex and $\Phi : T' \to T$ is a smooth embedding, then we have a pullback mapping $\Phi^* : C^\infty\Lambda^k(T) \to C^\infty\Lambda^k(T')$ for each $k \in \mathbb{Z}$. One can show that $\Phi^*\mathsf{d}^k\omega = \mathsf{d}^k\Phi^*\omega$ for each $\omega \in C^\infty\Lambda^k(T)$.

The trace operator $\operatorname{tr}^k_{T,F} : C^\infty\Lambda^k(T) \to C^\infty\Lambda^k(F)$ is defined as the pullback of k-forms along $\imath_{F,T}$. We have $\operatorname{tr}^k_{F,f}\operatorname{tr}^k_{T,F} = \operatorname{tr}^k_{T,f}$ for $F \in \Delta(T)$ and $f \in \Delta(F)$. Moreover, these traces are surjective.

We write $\{v^T_0,\dots,v^T_n\}$ for the set of vertices of T. The barycentric coordinates $\{\lambda^T_0,\dots,\lambda^T_n\}$ are the unique affine functions over T that satisfy $\lambda^T_i(v^T_j) = \delta_{ij}$ for $0 \leq i,j \leq n$. We introduce the barycentric monomials $\lambda^\alpha_T := \prod_{i=0}^n \left(\lambda^T_i\right)^{\alpha(i)}$ for $\alpha \in A(n)$, and define the space $\mathcal{P}_r(T)$ of barycentric polynomials up to order r as the span of barycentric monomials $\{\lambda^\alpha_T\}_{\alpha\in A(r,n)}$ of order r.

We define the barycentric k-alternators as the differential k-forms

$$\mathsf{d}\lambda^T_\sigma := \mathsf{d}\lambda^T_{\sigma(0)} \wedge \cdots \wedge \mathsf{d}\lambda^T_{\sigma(k)}, \quad \sigma \in \Sigma(1:k,0:n),$$

and the barycentric Whitney k-forms as

$$\phi^T_\rho := \sum_{i=0}^k (-1)\lambda^T_{\rho(i)}\mathsf{d}\lambda^T_{\rho-i}, \quad \rho \in \Sigma(0:k,0:n).$$

We then define

$$\mathcal{P}_r\Lambda^k(T) := \operatorname{span}\left\{ \lambda^\alpha_T\mathsf{d}\lambda^T_\sigma \;\middle|\; \alpha \in A(r,n),\ \sigma \in \Sigma(1:k,0:n) \right\}, \tag{3}$$

$$\mathcal{P}^-_r\Lambda^k(T) := \operatorname{span}\left\{ \lambda^\alpha_T\phi^T_\sigma \;\middle|\; \alpha \in A(r-1,n),\ \rho \in \Sigma(0:k,0:n) \right\}. \tag{4}$$

We adhere to the convention that $\mathcal{P}_r\Lambda^k(T) = \{0\}$ and $\mathcal{P}^-_r\Lambda^k(T) = \{0\}$ for negative polynomial order r. Note that $\mathcal{P}_r(T) = \mathcal{P}_r\Lambda^0(T)$.

If $F \in \Delta(T)$ is a subsimplex, then

$$\mathcal{P}_r\Lambda^k(F) = \operatorname{tr}^k_{T,F}\mathcal{P}_r\Lambda^k(T), \quad \mathcal{P}^-_r\Lambda^k(F) = \operatorname{tr}^k_{T,F}\mathcal{P}^-_r\Lambda^k(T). \tag{5}$$

Via the traces we define spaces with boundary conditions. We write

$$\mathring{\mathcal{P}}_r\Lambda^k(T) := \left\{\omega \in \mathcal{P}_r\Lambda^k(T) \;\middle|\; \forall F \in \Delta(T) : \operatorname{tr}^k_{T,F}\omega = 0\right\}, \tag{6}$$

$$\mathring{\mathcal{P}}^-_r\Lambda^k(T) := \left\{\omega \in \mathcal{P}^-_r\Lambda^k(T) \;\middle|\; \forall F \in \Delta(T) : \operatorname{tr}^k_{T,F}\omega = 0\right\}. \tag{7}$$

These spaces are affinely invariant in the following sense. For every bijective affine mapping $\Phi : T' \to T$ from a simplex T' onto T we have

$$\mathcal{P}_r\Lambda^k(T') = \phi^*\mathcal{P}_r\Lambda^k(T), \quad \mathcal{P}_r^-\Lambda^k(T') = \phi^*\mathcal{P}_r^-\Lambda^k(T), \tag{8}$$

$$\mathring{\mathcal{P}}_r\Lambda^k(T') = \phi^*\mathring{\mathcal{P}}_r\Lambda^k(T), \quad \mathring{\mathcal{P}}_r^-\Lambda^k(T') = \phi^*\mathring{\mathcal{P}}_r^-\Lambda^k(T). \tag{9}$$

We recall some further inclusions and identities. One can show that

$$\mathcal{P}_r\Lambda^k(T) \subseteq \mathcal{P}_{r+1}^-\Lambda^k(T) \subseteq \mathcal{P}_{r+1}\Lambda^k(T), \tag{10}$$

$$\mathring{\mathcal{P}}_r\Lambda^k(T) \subseteq \mathring{\mathcal{P}}_{r+1}^-\Lambda^k(T) \subseteq \mathring{\mathcal{P}}_{r+1}\Lambda^k(T), \tag{11}$$

$$\mathsf{d}^k\mathcal{P}_r\Lambda^k(T) \subseteq \mathcal{P}_{r-1}\Lambda^{k+1}(T), \quad \mathsf{d}^k\mathring{\mathcal{P}}_r\Lambda^k(T) \subseteq \mathring{\mathcal{P}}_{r-1}\Lambda^{k+1}(T), \tag{12}$$

$$\mathsf{d}^k\mathcal{P}_r^-\Lambda^k(T) = \mathsf{d}^k\mathcal{P}_r\Lambda^k(T), \quad \mathsf{d}^k\mathring{\mathcal{P}}_r^-\Lambda^k(T) = \mathsf{d}^k\mathring{\mathcal{P}}_r\Lambda^k(T). \tag{13}$$

For positive polynomial order $r \geq 1$ we additionally have

$$\mathcal{P}_r^-\Lambda^0(T) = \mathcal{P}_r\Lambda^0(T), \quad \mathring{\mathcal{P}}_r^-\Lambda^0(T) = \mathring{\mathcal{P}}_r\Lambda^0(T), \tag{14}$$

$$\mathcal{P}_r^-\Lambda^n(T) = \mathcal{P}_{r-1}\Lambda^n(T), \quad \mathring{\mathcal{P}}_r^-\Lambda^n(T) = \mathring{\mathcal{P}}_{r-1}\Lambda^n(T). \tag{15}$$

It has been established by (4) that $\mathring{\mathcal{P}}_r\Lambda^k = \{0\}$ if $r < n-k-1$ and that $\mathring{\mathcal{P}}_r^-\Lambda^k = \{0\}$ if $r < n-k+1$. This follows, for example, from the dimension countings

$$\dim \mathcal{P}_r\Lambda^k(T) = \binom{n+r}{n}\binom{n}{k}, \quad \dim \mathcal{P}_r^-\Lambda^k(T) = \binom{r+k-1}{k}\binom{n+r}{n-k},$$
$$\dim \mathring{\mathcal{P}}_r\Lambda^k(T) = \binom{r+1}{n-k}\binom{r+k}{k}, \quad \dim \mathring{\mathcal{P}}_r^-\Lambda^k(T) = \binom{n}{k}\binom{r+k-1}{n}.$$

Remark 2.1.
Definitions (3) and (4) of $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}_r^-\Lambda^k(T)$ are in terms of spanning sets that are not linearly independent in general. For explicit bases for $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}_r^-\Lambda^k(T)$ and the spaces $\mathring{\mathcal{P}}_r\Lambda^k(T)$ and $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ we refer to (5).

Two important polynomial differential forms over T are $1_T \in \mathcal{P}_0\Lambda^0(T)$, the constant function over T which at each point takes the value 1, and the volume form $\mathrm{vol}_T \in \mathcal{P}_0\Lambda^n(T)$, the unique constant n-form over T with $\int_T \mathrm{vol}_T = \mathrm{vol}^n(T)$. These span the constant functions $\mathcal{P}_0\Lambda^0(T)$ and the constant n-forms $\mathcal{P}_0\Lambda^n(T)$, respectively. The former is the kernel of d^0 and the latter is complementary to the range of d^{n-1}. It will be convenient to introduce notation for spaces with those special differential forms removed. Let $\int_T : C^\infty\Lambda^0(T) \to \mathbb{R}$ and $\int_T : C^\infty\Lambda^n(T) \to \mathbb{R}$ denote the respective integral mappings of 0- and n-forms over T. We set

$$\underline{\mathcal{P}}_r\Lambda^k(T) := \begin{cases} \mathcal{P}_r\Lambda^0(T) \cap \ker \int_T & \text{if } k = 0, \\ \mathcal{P}_r\Lambda^k(T) & \text{otherwise,} \end{cases} \tag{17}$$

$$\underline{\mathcal{P}}_r^-\Lambda^k(T) := \begin{cases} \mathcal{P}_r\Lambda^0(T) \cap \ker \int_T & \text{if } k = 0, \\ \mathcal{P}_r^-\Lambda^k(T) & \text{otherwise,} \end{cases} \tag{18}$$

$$\underline{\mathring{\mathcal{P}}}_r\Lambda^k(T) := \begin{cases} \mathring{\mathcal{P}}_r\Lambda^n(T) \cap \int_T & \text{if } k = n, \\ \mathring{\mathcal{P}}_r\Lambda^k(T) & \text{otherwise,} \end{cases} \tag{19}$$

$$\underline{\mathring{\mathcal{P}}}_r^-\Lambda^k(T) := \begin{cases} \mathring{\mathcal{P}}_r^-\Lambda^n(T) \cap \int_T & \text{if } k = n, \\ \mathring{\mathcal{P}}_r^-\Lambda^k(T) & \text{otherwise.} \end{cases} \tag{20}$$

We obviously have for $r \geq 0$ the direct sum decompositions

$$\mathcal{P}_r\Lambda^0(T) = \underline{\mathcal{P}}_r\Lambda^0(T) \oplus \mathbb{R}\cdot 1_T, \quad \mathcal{P}^-_{r+1}\Lambda^0(T) = \underline{\mathcal{P}}^-_{r+1}\Lambda^0(T) \oplus \mathbb{R}\cdot 1_T, \tag{21}$$

$$\mathring{\mathcal{P}}_r\Lambda^n(T) = \mathring{\underline{\mathcal{P}}}_r\Lambda^n(T) \oplus \mathbb{R}\cdot \mathrm{vol}_T, \quad \mathring{\mathcal{P}}^-_{r+1}\Lambda^n(T) = \mathring{\underline{\mathcal{P}}}^-_{r+1}\Lambda^n(T) \oplus \mathbb{R}\cdot \mathrm{vol}_T, \tag{22}$$

and no changes in the other cases.

With these spaces, we may concisely state the following exactness properties of polynomial differential forms which have been proven by (4). One can show

$$\forall \omega \in \underline{\mathcal{P}}_r\Lambda^k(T) : \left(\mathsf{d}^k\omega = 0 \implies \exists \eta \in \underline{\mathcal{P}}^-_{r+1}\Lambda^{k-1} : \mathsf{d}^{k-1}\eta = \omega\right), \tag{23}$$

$$\forall \omega \in \mathring{\underline{\mathcal{P}}}_r\Lambda^k(T) : \left(\mathsf{d}^k\omega = 0 \implies \exists \eta \in \mathring{\underline{\mathcal{P}}}^-_{r+1}\Lambda^{k-1} : \mathsf{d}^{k-1}\eta = \omega\right). \tag{24}$$

These results will be used in the next section.

Example 2.2.
We recapitulate a few examples how these concepts translate to classical finite element spaces when T is a triangle and $r \geq 1$. We refer to (4) for further elaboration on these examples.

For $k = 0$ the space $\mathcal{P}_r\Lambda^0(T) = \mathcal{P}^-_r\Lambda^0(T)$ translates into the space of order r polynomials over T. Additionally $\mathring{\mathcal{P}}_r\Lambda^0(T) = \mathring{\mathcal{P}}^-_r\Lambda^0(T)$ is the subspace satisfying Dirichlet boundary conditions along the edges, and $\underline{\mathcal{P}}_r\Lambda^0(T) = \underline{\mathcal{P}}^-_r\Lambda^0(T)$ is the subspace of order r polynomials with vanishing mean value.

For $k = 1$ the two families translate into different spaces: $\mathcal{P}_r\Lambda^k(T)$ translates into the order r Brezzi-Douglas-Marini space $\mathbf{BDM}_r(T)$ and $\mathcal{P}^-_r\Lambda^k(T)$ translates into the order r Raviart-Thomas space $\mathbf{RT}_r(T)$. We write $\mathring{\mathbf{BDM}}_r(T)$ and $\mathring{\mathbf{RT}}_r(T)$ for the subspaces with boundary conditions, which in this case are normal boundary conditions along the simplex boundary.

Finally, for $k = 2$ we have $\mathcal{P}_{r-1}\Lambda^2(T) = \mathcal{P}^-_r\Lambda^2(T)$. This space translates into polynomials over T of order $r-1$, but this time imposing boundary conditions does not change the space. The subspace $\underline{\mathcal{P}}_{r-1}\Lambda^2(T) = \underline{\mathcal{P}}^-_r\Lambda^2(T)$ corresponds to the order $(r-1)$ polynomials over T with vanishing mean value.

3. Polynomial de Rham Complexes over Simplices

This section develops a theory of polynomial de Rham complexes over simplices. We prove their exactness and obtain a representation of the degrees of freedom. We first observe that differential complexes of similar *type* appear throughout finite element exterior calculus in different variants. For example, a differential complex of trimmed polynomial differential forms of fixed order r appears as differential complex over a single simplex, over a triangulation, or with boundary conditions. It is of interest to turn the idea of sequences having a *type* into a rigorous mathematical notion. A particular motivation are differential complexes in the theory of *hp*-adaptive methods, composed of finite element spaces of non-uniform polynomial order. In that application we wish to assign types of polynomial de Rham complexes to each simplex to describe the local order of approximation.

We first introduce a set of formal symbols

$$\mathscr{S} := \left\{\ldots, \mathcal{P}_{r-1}, \mathcal{P}^-_r, \mathcal{P}_r, \mathcal{P}^-_{r+1}, \ldots\right\}. \tag{25}$$

The set $\mathscr{S}$ is endowed with a total order $\leq$ defined by $\mathcal{P}^-_r \leq \mathcal{P}_r$ and $\mathcal{P}_r \leq \mathcal{P}^-_{r+1}$.

An *admissible sequence type* is a mapping $\mathcal{P} : \mathbb{Z} \to \mathscr{S}$ that satisfies the condition

$$\forall k \in \mathbb{Z} : \mathcal{P}(k) \in \left\{ \mathcal{P}_r^-, \mathcal{P}_r \right\} \implies \mathcal{P}(k+1) \in \left\{ \mathcal{P}_r^-, \mathcal{P}_{r-1} \right\}. \tag{26}$$

If $\mathcal{P} \in \mathscr{A}$ is an admissible sequence type and T is an n-simplex, then we define for each $k \in \mathbb{Z}$ the spaces

$$\mathcal{P}\Lambda^k(T) := \begin{cases} \mathcal{P}_r\Lambda^k(T) & \text{if } \mathcal{P}(k) = \mathcal{P}_r, \\ \mathcal{P}_r^-\Lambda^k(T) & \text{if } \mathcal{P}(k) = \mathcal{P}_r^-, \end{cases} \tag{27}$$

$$\mathring{\mathcal{P}}\Lambda^k(T) := \begin{cases} \mathring{\mathcal{P}}_r\Lambda^k(T) & \text{if } \mathcal{P}(k) = \mathcal{P}_r, \\ \mathring{\mathcal{P}}_r^-\Lambda^k(T) & \text{if } \mathcal{P}(k) = \mathcal{P}_r^-, \end{cases} \tag{28}$$

$$\underline{\mathcal{P}}\Lambda^k(T) := \begin{cases} \underline{\mathcal{P}}_r\Lambda^k(T) & \text{if } \mathcal{P}(k) = \mathcal{P}_r, \\ \underline{\mathcal{P}}_r^-\Lambda^k(T) & \text{if } \mathcal{P}(k) = \mathcal{P}_r^-, \end{cases} \tag{29}$$

$$\mathring{\underline{\mathcal{P}}}\Lambda^k(T) := \begin{cases} \mathring{\underline{\mathcal{P}}}_r\Lambda^k(T) & \text{if } \mathcal{P}(k) = \mathcal{P}_r, \\ \mathring{\underline{\mathcal{P}}}_r^-\Lambda^k(T) & \text{if } \mathcal{P}(k) = \mathcal{P}_r^-. \end{cases} \tag{30}$$

We let $\mathscr{A}$ denote the set of admissible sequence types. The total order on $\mathscr{S}$ induces a partial order $\leq$ on $\mathscr{A}$, where for all $\mathcal{P}, \mathcal{S} \in \mathscr{A}$ we have $\mathcal{P} \leq \mathcal{S}$ if and only if for all $k \in \mathbb{Z}$ we have $\mathcal{P}(k) \leq \mathcal{S}(k)$.

The notation already suggests that the symbols $\mathscr{S}$ describe finite element spaces, whereas the admissible sequence types $\mathscr{A}$ describe finite element differential complexes. To make this idea rigorous, we begin with an easy observation that follows from (26). For each admissible sequence type $\mathcal{P} \in \mathscr{A}$, $k \in \mathbb{Z}$ and simplex T we have

$$\begin{aligned} \mathsf{d}^k\mathcal{P}\Lambda^k(T) &\subseteq \mathcal{P}\Lambda^{k+1}(T), \quad & \mathsf{d}^k\mathring{\mathcal{P}}\Lambda^k(T) &\subseteq \mathring{\mathcal{P}}\Lambda^{k+1}(T), \\ \mathsf{d}^k\underline{\mathcal{P}}\Lambda^k(T) &\subseteq \underline{\mathcal{P}}\Lambda^{k+1}(T), \quad & \mathsf{d}^k\mathring{\underline{\mathcal{P}}}\Lambda^k(T) &\subseteq \mathring{\underline{\mathcal{P}}}\Lambda^{k+1}(T). \end{aligned}$$

In the light of this we compose differential complexes in accordance with a given admissible sequence type. Suppose that T is a simplex and that $\mathcal{P} \in \mathscr{A}$ is an admissible sequence type. Then we have a polynomial de Rham complex over T,

$$0 \to \mathbb{R} \longrightarrow \mathcal{P}\Lambda^0(T) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \mathcal{P}\Lambda^n(T) \to 0 \tag{31}$$

and a polynomial de Rham complex over T with boundary conditions,

$$0 \to \mathring{\mathcal{P}}\Lambda^0(T) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \mathring{\mathcal{P}}\Lambda^n(T) \longrightarrow \mathbb{R} \to 0. \tag{32}$$

We will also consider the reduced differential complexes

$$0 \to \underline{\mathcal{P}}\Lambda^0(T) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \underline{\mathcal{P}}\Lambda^n(T) \to 0 \tag{33}$$

$$0 \to \mathring{\underline{\mathcal{P}}}\Lambda^0(T) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \mathring{\underline{\mathcal{P}}}\Lambda^n(T) \to 0. \tag{34}$$

We establish the exactness of these differential complexes.

Lemma 3.1.
Let T be a simplex and let $\mathcal{P} \in \mathscr{A}$ be an admissible sequence type. If $1_T \in \mathcal{P}\Lambda^0(T)$, then (31) is well-defined and exact. If $\mathrm{vol}_T \in \mathring{\mathcal{P}}\Lambda^n(T)$, then (32) is exact.

Proof. With regards to the first sequence, it is obvious that $\ker \mathsf{d}^0 \cap \mathcal{P}\Lambda^0(T)$ is spanned by 1_T. Let $k \in \{1, \dots, n\}$ and $\omega \in \mathcal{P}\Lambda^k(T)$ with $\mathsf{d}^k\omega = 0$. Then there exists $r \in \mathbb{Z}$ with $\omega \in \mathcal{P}_r\Lambda^k(T)$. By Lemma 3.8 of (4) there exists $\xi \in \mathcal{P}_{r+1}^-\Lambda^{k-1}(T)$ with $\mathsf{d}^{k-1}\xi = \omega$. Since $\mathcal{P}_{r+1}^-\Lambda^{k-1}(T) \subseteq \mathcal{P}\Lambda^{k-1}(T)$, the exactness of the first sequence follows.

With regards to the second sequence, it is obvious that $\ker \mathsf{d}^0 \cap \mathring{\mathcal{P}}\Lambda^0(T)$ is the trivial vector space. Now let $k \in \{1,\dots,n\}$ and $\omega \in \mathcal{P}\Lambda^k(T)$ with $\mathsf{d}^k\omega = 0$. We assume additionally $\int_T \omega = 0$ if $k = n$. There exists $r \in \mathbb{Z}$ such that $\omega \in \mathring{\mathcal{P}}_r\Lambda^k(T)$. Using the smoothed projection of (20) over a single simplex with full boundary conditions, it is easy to prove the existence of $\eta \in \mathring{\mathcal{P}}^-_{r+1}\Lambda^{k-1}(T)$ with $\mathsf{d}^{k-1}\eta = \omega$. But we also have $\mathring{\mathcal{P}}^-_{r+1}\Lambda^{k-1}(T) \subseteq \mathring{\mathcal{P}}\Lambda^{k-1}(T)$. This completes the proof. □

Lemma 3.2.
Let T be a simplex and let $\mathcal{P}$ be an admissible sequence type. Then (33) and (34) are exact sequences. □

Proof. If $1_T \in \mathcal{P}\Lambda^0(T)$, then $\mathcal{P}\Lambda^0(T) = \mathbb{R}\cdot 1_T \oplus \underline{\mathcal{P}}\Lambda^0(T)$, and if $\mathrm{vol}_T \in \mathring{\mathcal{P}}\Lambda^n(T)$, then $\mathcal{P}\Lambda^n(T) = \mathbb{R}\cdot \mathrm{vol}_T \oplus \mathring{\underline{\mathcal{P}}}\Lambda^n(T)$. The claim now follows immediately from the preceding result. □

Now we move our attention towards dual spaces and their representations. This prepares the discussion of the degrees of freedom of finite element de Rham complexes in later sections. Our approach to the degrees of freedom differs from the approach of (4) but is inspired by (23).

Let T be a simplex and let g be a smooth Riemannian metric over T. This induces a positive definite bilinear form (see 26)

$$B_g : C^\infty\Lambda^k(T) \times C^\infty\Lambda^k(T) \to \mathbb{R}, \quad (\omega,\eta) \mapsto \int_T \langle \omega, \eta\rangle_g.$$

The restriction of this bilinear form to any finite-dimensional subspace of $C^\infty\Lambda^k(T)$ gives a Hilbert space structure on that subspace. We apply this idea to the spaces $\mathring{\underline{\mathcal{P}}}\Lambda^k(T)$, since this is the special case needed in later sections. The following lemma, however, can be generalized to the spaces of the form $\mathcal{P}\Lambda^k(T)$, $\mathring{\mathcal{P}}\Lambda^k(T)$ and $\underline{\mathcal{P}}\Lambda^k(T)$ with minimal changes.

Lemma 3.3.
Let $\mathcal{P} \in \mathscr{A}$. Let $\Psi : \mathring{\underline{\mathcal{P}}}\Lambda^k(T) \to \mathbb{R}$ be a linear functional. Then there exist $\rho \in \mathring{\underline{\mathcal{P}}}\Lambda^{k-1}(T)$ and $\beta \in \mathring{\underline{\mathcal{P}}}\Lambda^k(T)$ such that

$$\Psi(\omega) = \int_T \langle \omega, \mathsf{d}^{k-1}\rho\rangle_g + \int_T \langle \mathsf{d}^k\omega, \mathsf{d}^k\beta\rangle_g, \quad \omega \in \mathring{\underline{\mathcal{P}}}\Lambda^k(T).$$

Proof. Let $\Psi : \mathring{\underline{\mathcal{P}}}\Lambda^k(T) \to \mathbb{R}$ be linear and let $\omega \in \mathring{\underline{\mathcal{P}}}\Lambda^k(T)$ be arbitrary. Since B_g induces a Hilbert space structure on a finite-dimensional vector space, the Riesz representation theorem ensures the existence of $\eta \in \mathring{\underline{\mathcal{P}}}\Lambda^k(T)$ such that $\Psi(\omega) = B_g(\omega,\eta)$. We write $A_0 = \mathring{\underline{\mathcal{P}}}\Lambda^k(T) \cap \ker \mathsf{d}^k$ and let A_1 denote the orthogonal complement of A_0 in $\mathring{\underline{\mathcal{P}}}\Lambda^k(T)$ with respect to the scalar product B_g. We have an orthogonal decomposition $\mathring{\underline{\mathcal{P}}}\Lambda^k(T) = A_0 \oplus A_1$, and unique decomposition $\omega = \omega_0 + \omega_1$ and $\eta = \eta_0 + \eta_1$ with $\omega_0, \eta_0 \in A_0$ and $\omega_1, \eta_1 \in A_1$. Thus

$$\Psi(\omega) = \int_T \langle \omega, \eta\rangle = \int_T \langle \omega_0, \eta_0\rangle + \int_T \langle \omega_1, \eta_1\rangle.$$

By the exactness of (34) there exists $\rho \in \mathring{\underline{\mathcal{P}}}\Lambda^{k-1}(T)$ such that $\eta_0 = \mathsf{d}^{k-1}\rho$. Since the bilinear form $B_g\left(\mathsf{d}^k\cdot, \mathsf{d}^k\cdot\right)$ is a scalar product over A_1 equivalent to B_g, we may use the Riesz representation theorem again to obtain $\beta \in \mathring{\underline{\mathcal{P}}}\Lambda^k(T)$ with $B_g\left(\mathsf{d}^k\omega_1, \mathsf{d}^k\beta\right) = B_g\left(\omega_1, \eta_1\right)$. The proof is complete. □

4. The Complex of Whitney Forms

In the preceding section we have studied finite element differential complexes over simplices. We now proceed to finite element differential complexes over triangulations. We begin in this section with the special case of lowest order: the complexes of Whitney forms. An important concept is the canonical interpolator.

Let $\mathcal{T}$ be a simplicial complex. This means that $\mathcal{T}$ is a set of simplices such that

$$\forall T \in \mathcal{T} : \forall F \in \Delta(T) : F \in \mathcal{T}, \tag{35a}$$

$$\forall T, T' \in \mathcal{T} : T \cap T' \in \mathcal{T} \cup \{\emptyset\}. \tag{35b}$$

In other words, the set of simplices $\mathcal{T}$ is closed under taking subsimplices and the intersection of two simplices in $\mathcal{T}$ is either empty or a common subsimplex. We let $\mathcal{T}^k$ denote the set of k-simplices in $\mathcal{T}$. The simplest example of a simplicial complex is the set of subsimplices $\Delta(T)$ of any simplex T. Other examples are triangulations of domains. A simplicial complex $\mathcal{U} \subseteq \mathcal{T}$ is called a simplicial subcomplex of $\mathcal{T}$. Note that $\mathcal{U} = \emptyset$ is possible.

For each triangulation we have an associated simplicial chain complex. We recall that we assume the simplices in $\mathcal{T}$ to be equipped with an arbitrary but fixed orientation. The space $\mathcal{C}_k(\mathcal{T})$ of simplicial k-chains is defined as the real vector space spanned by the oriented k-simplices in $\mathcal{T}^k$.

We recall that the orientation of a simplex T induces an orientation on its subsimplices of one dimension lower. When $T \in \mathcal{T}^k$ and $F \in \mathcal{T}^{k-1}$ with $F \in \Delta(T)$, then we set $o(F,T) := 1$ if the fixed orientation over T induces the fixed orientation over F, and set $o(F,T) := -1$ in the opposite case. The *simplicial boundary operator* is the linear operator

$$\partial_k : \mathcal{C}_k(\mathcal{T}) \to \mathcal{C}_{k-1}(\mathcal{T})$$

that is defined by taking the linear extension of setting

$$\partial_k T := \sum_{F \in \Delta(T)^{k-1}} o(F,T) F, \quad T \in \mathcal{T}^k.$$

This operator satisfies the differential property $\partial_{k-1}\partial_k = 0$. When $\mathcal{U} \subseteq \mathcal{T}$ is a simplicial subcomplex then we define the vector space $\mathcal{C}_k(\mathcal{T},\mathcal{U})$ as the factor space

$$\mathcal{C}_k(\mathcal{T},\mathcal{U}) := \mathcal{C}_k(\mathcal{T})/\mathcal{C}_k(\mathcal{U}).$$

Note that $\mathcal{C}_k(\mathcal{T},\emptyset) = \mathcal{C}_k(\mathcal{T})$. A canonical basis of $\mathcal{C}_k(\mathcal{T},\mathcal{U})$ is given by (the equivalence classes of) the oriented k-simplices in $\mathcal{T}^k$ which are not contained in $\mathcal{U}^k$. In particular, we can identify $\mathcal{C}_k(\mathcal{T},\mathcal{U})$ with the subspace of $\mathcal{C}_m(\mathcal{T})$ spanned by $\mathcal{T}^k \setminus \mathcal{U}^k$. The simplicial boundary operator induces a well-defined operator

$$\partial_k : \mathcal{C}_k(\mathcal{T},\mathcal{U}) \to \mathcal{C}_{k-1}(\mathcal{T},\mathcal{U}),$$

which again satisfies the differential property $\partial_{k-1}\partial_k = 0$. Accordingly we introduce the simplicial chain complex

$$\dots \xleftarrow{\partial_{k-1}} \mathcal{C}_{k-1}(\mathcal{T},\mathcal{U}) \xleftarrow{\partial_k} \mathcal{C}_k(\mathcal{T},\mathcal{U}) \xleftarrow{\partial_{k+1}} \dots \tag{36}$$

The dimension of the k-th homology space of this complex,

$$b_k(\mathcal{T},\mathcal{U}) := \dim \frac{\ker \partial_k : \mathcal{C}_k(\mathcal{T},\mathcal{U}) \to \mathcal{C}_{k-1}(\mathcal{T},\mathcal{U})}{\operatorname{ran} \partial_{k+1} : \mathcal{C}_{k+1}(\mathcal{T},\mathcal{U}) \to \mathcal{C}_k(\mathcal{T},\mathcal{U})}, \tag{37}$$

is known as the *k-th simplicial Betti number* of $\mathcal{T}$ relative to $\mathcal{U}$. If $\mathcal{U} = \emptyset$, then we call $b_k(\mathcal{T}) := b_k(\mathcal{T},\mathcal{U})$ just the *k-th simplicial Betti number* of $\mathcal{T}$.

We now introduce differential forms into the discussion. We define

$$C^\infty\Lambda^k(\mathcal{T}) := \left\{ (\omega_T)_T \in \bigoplus_{T\in\mathcal{T}} C^\infty\Lambda^k(T) \;\middle|\; \forall T \in \mathcal{T} : \forall F \in \Delta(T) : \operatorname{tr}^k_{T,F} \omega_T = \omega_F \right\}.$$

Via a linear algebraic isomorphism we may identify the space $C^\infty\Lambda^k(\mathcal{T})$ with the space of differential k-forms that are piecewise smooth with respect to $\mathcal{T}$ and that have single-valued traces along simplex boundaries. Our choice of formalism will simplify the notation in the sequel. Henceforth, we may also write $\operatorname{tr}_T \omega := \omega_T$ for $\omega \in C^\infty\Lambda^k(\mathcal{T})$ and $T \in \mathcal{T}$.

Because the exterior derivative commutes with trace operators, we have a well-defined exterior derivative on $C^\infty\Lambda^k(\mathcal{T})$ given by

$$\mathsf{d}^k : C^\infty\Lambda^k(\mathcal{T}) \to C^\infty\Lambda^{k+1}(\mathcal{T}), \quad (\omega_T)_{T\in\mathcal{T}} \mapsto (\mathsf{d}^k\omega_T)_{T\in\mathcal{T}}. \tag{38}$$

Since $\mathsf{d}^{k+1}\mathsf{d}^k\omega = 0$ for every $\omega \in C^\infty\Lambda^k(\mathcal{T})$, we may compose a differential complex

$$\dots \xrightarrow{\mathsf{d}^{k-1}} C^\infty\Lambda^k(\mathcal{T}) \xrightarrow{\mathsf{d}^k} C^\infty\Lambda^{k+1}(\mathcal{T}) \xrightarrow{\mathsf{d}^{k+1}} \dots \tag{39}$$

In order to formalize boundary conditions, we furthermore define

$$C^\infty\Lambda^k(\mathcal{T},\mathcal{U}) := \left\{ \omega \in C^\infty\Lambda^k(\mathcal{T}) \;\middle|\; \forall F \in \mathcal{U} : \omega_F = 0 \right\}. \tag{40}$$

It is easily verified that

$$\mathsf{d}^k \left(C^\infty\Lambda^k(\mathcal{T},\mathcal{U}) \right) \subseteq C^\infty\Lambda^{k+1}(\mathcal{T},\mathcal{U}). \tag{41}$$

In particular, we may compose the differential complex

$$\dots \xrightarrow{\mathsf{d}^{k-1}} C^\infty\Lambda^k(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^k} C^\infty\Lambda^{k+1}(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^{k+1}} \dots \tag{42}$$

with abstract boundary conditions.

Remark 4.1.
Constructions similar to our definition of $C^\infty\Lambda^k(\mathcal{T})$ have appeared in mathematics before. Our definition is a special case of a *finite element system* in the terminology of (19). Another variant is exemplified by *Sullivan forms* in global analysis (see 24), which are piecewise *flat* differential forms in the sense of geometric measure theory.

For a practical illustration, suppose that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain triangulated by a simplicial complex $\mathcal{T}$. Then the members of $C^\infty\Lambda^k(\mathcal{T})$ correspond to the differential k-forms over Ω that are piecewise smooth with respect to $\mathcal{T}$ and have single-valued traces on subsimplices.

Suppose that $\Gamma \subset \partial\Omega$ is a subset of the boundary and that $\mathcal{U}$ is a simplicial subcomplex of $\mathcal{T}$ triangulating Γ. Then $C^\infty\Lambda^k(\mathcal{T},\mathcal{U})$ is the subspace of $C^\infty\Lambda^k(\mathcal{T},\mathcal{U})$ whose member have vanishing traces along Γ. In this manner $\mathcal{U}$ may be used to model homogeneous boundary conditions.

We investigate an important relation between the simplicial chains and the piecewise smooth differential forms with respect to $\mathcal{T}$. Suppose that $\omega \in C^\infty\Lambda^k(\mathcal{T},\mathcal{U})$ and $T \in \mathcal{T}^k \setminus \mathcal{U}^k$. We then write $\int_T \omega := \int_T \operatorname{tr}^k_T \omega_T$ for the integral of ω over T. By linear extension we obtain a bilinear pairing

$$C^\infty\Lambda^k(\mathcal{T},\mathcal{U}) \times \mathcal{C}_k(\mathcal{T},\mathcal{U}) \to \mathbb{R}, \quad (\omega, S) \to \int_S \omega. \tag{43}$$

Moreover we easily observe (by first considering a single simplex and then taking the linear extension) that

$$\int_{\partial_{k+1} S} \omega = \int_S \mathsf{d}^k\omega, \quad \omega \in C^\infty\Lambda^k(\mathcal{T},\mathcal{U}), \quad S \in \mathcal{C}_{k+1}(\mathcal{T},\mathcal{U}).$$

The linear pairing (43) is degenerate in general.

We will identify a differential subcomplex of (42) restricting to which in the first variable makes the bilinear pairing (43) non-degenerate. Specifically, we employ a finite element de Rham complex of lowest polynomial order. To begin with, we define the spaces of *Whitney forms* by

$$\mathcal{W}\Lambda^k(\mathcal{T}) := \left\{ \omega \in C^\infty\Lambda^k(\mathcal{T}) \mid \forall T \in \mathcal{U} : \omega_T \in \mathcal{P}_1^-\Lambda^k(T) \right\},$$
$$\mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}) := \mathcal{W}\Lambda^k(\mathcal{T}) \cap C^\infty\Lambda^k(\mathcal{T},\mathcal{U}).$$

It is an immediate consequence of definitions that we have a well-defined operator

$$\mathsf{d}^k : \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}) \to \mathcal{W}\Lambda^{k+1}(\mathcal{T},\mathcal{U}),$$

and consequently the *differential complex of Whitney forms*

$$\dots \xrightarrow{\mathsf{d}^{k-1}} \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^k} \mathcal{W}\Lambda^{k+1}(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^{k+1}} \dots \tag{44}$$

The notion of Whitney forms was originally motivated by their duality to the simplicial chains. This is summarized in the following lemma, which has been proven many times (17; 19).

Lemma 4.2.
The bilinear pairing

$$\mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}) \times \mathcal{C}_k(\mathcal{T},\mathcal{U}) \to \mathbb{R}, \quad (\omega, S) \mapsto \int_S \operatorname{tr}^k_S \omega \tag{45}$$

is non-degenerate. □

As a consequence of Lemma 4.2 we obtain a linear isomorphism between $\mathcal{C}^k(\mathcal{T},\mathcal{U})$ and the dual space of $\mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U})$,

$$\mathcal{C}_k(\mathcal{T},\mathcal{U}) \simeq \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U})'.$$

In particular, the differential complex of simplicial chains (36) is isomorphic to the dual complex of the complex of Whitney forms (44), and the simplicial boundary operator $\partial_{k+1} : \mathcal{C}_{k+1}(\mathcal{T},\mathcal{U}) \to \mathcal{C}_k(\mathcal{T},\mathcal{U})$ is isomorphic to the dual operator of $\mathsf{d}^k : \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}) \to \mathcal{W}\Lambda^{k+1}(\mathcal{T},\mathcal{U})$. One can now show that the cohomology spaces of the complex of Whitney forms (44) have the same dimension as the cohomology spaces as the corresponding cohomology spaces of the simplicial chain complex (36). This dimension is precisely the simplicial Betti number $b_k(\mathcal{T},\mathcal{U})$. In summary,

$$\dim \frac{\ker \mathsf{d}^k : \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}) \to \mathcal{W}\Lambda^{k+1}(\mathcal{T},\mathcal{U})}{\operatorname{ran} \mathsf{d}^{k-1} : \mathcal{W}\Lambda^{k-1}(\mathcal{T},\mathcal{U}) \to \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U})} = b_k(\mathcal{T},\mathcal{U}), \tag{46}$$

as follows from (37).

Remark 4.3.
Whitney forms are discussed in Whitney's monograph on geometric measure theory (40). They have received attention in numerical analysis for almost 30 years (11; 28).

We are now in a position to provide the canonical finite element interpolator from the space $C^\infty\Lambda^k(\mathcal{T})$ onto the space $\mathcal{W}\Lambda^k(\mathcal{T})$. We define

$$I^k_{\mathcal{W}} : C^\infty\Lambda^k(\mathcal{T}) \to \mathcal{W}\Lambda^k(\mathcal{T}) \tag{47}$$

by setting

$$\int_S I^k_{\mathcal{W}}\omega = \int_S \omega, \quad \omega \in C^\infty\Lambda^k(\mathcal{T}), \quad S \in \mathcal{C}_k(\mathcal{T}).$$

This is well-defined because of Lemma 4.2.

The operator $I^k_{\mathcal{W}}$ acts as the identity on Whitney forms, i.e.

$$I^k_{\mathcal{W}}\omega = \omega, \quad \omega \in \mathcal{W}\Lambda^k(\mathcal{T}).$$

The operator $I^k_{\mathcal{W}}$ is local in the sense that for every $C \in \mathcal{T}$ we have

$$\omega_C = 0 \implies (I^k_{\mathcal{W}}\omega)_C = 0.$$

By restricting the interpolant to $C^\infty\Lambda^k(\mathcal{T},\mathcal{U})$ we obtain a well-defined mapping

$$I^k_{\mathcal{W}} : C^\infty\Lambda^k(\mathcal{T},\mathcal{U}) \to \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}).$$

The interpolation operator commutes with the exterior derivative,

$$\mathsf{d}^k I^k_{\mathcal{W}}\omega = I^{k+1}_{\mathcal{W}}\mathsf{d}^k\omega, \quad \omega \in C^\infty\Lambda^k(\mathcal{T}),$$

as we verify by

$$\int_S I^{k+1}_{\mathcal{W}}\mathsf{d}^k\omega = \int_S \mathsf{d}^k\omega = \int_{\partial_{k+1}S} \omega = \int_{\partial_{k+1}S} I^k_{\mathcal{W}}\omega = \int_S \mathsf{d}^k I^k_{\mathcal{W}}\omega$$

for $S \in \mathcal{C}_{k+1}(\mathcal{T})$ and $\omega \in C^\infty\Lambda^k(\mathcal{T})$. So the diagram

$$\begin{array}{ccccccc}
\dots & \xrightarrow{\mathsf{d}^{k-1}} & C^\infty\Lambda^k(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^k} & C^\infty\Lambda^{k+1}(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^{k+1}} & \dots \\
 & & \downarrow{\scriptstyle I^k_{\mathcal{W}}} & & \downarrow{\scriptstyle I^{k+1}_{\mathcal{W}}} & & \\
\dots & \xrightarrow{\mathsf{d}^{k-1}} & \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^k} & \mathcal{W}\Lambda^{k+1}(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^{k+1}} & \dots
\end{array}$$

commutes. In particular, $I^k_{\mathcal{W}}$ is a morphism of differential complexes.

5. Higher Order Finite Element Complexes

In this section we study the structure of higher order finite element differential complexes over triangulations. We also construct a global interpolant. This combines the theoretical preparations carried out in Section 3 and Section 4.

Let $\mathcal{T}$ be a simplicial complex and let $\mathcal{U}$ be a (possibly empty) subcomplex of $\mathcal{T}$. We let $\mathcal{P} : \mathcal{T} \to \mathscr{A}$ be a mapping that associates to each simplex $T \in \mathcal{T}$ an admissible sequence type $\mathcal{P}_T : \mathbb{Z} \to \mathscr{A}$. We then define

$$\mathcal{P}\Lambda^k(\mathcal{T}) := \left\{ \omega \in C^\infty\Lambda^k(\mathcal{T}) \mid \forall T \in \mathcal{T} : \omega_T \in \mathcal{P}_T\Lambda^k(T) \right\}. \tag{48}$$

By construction, the exterior derivative preserves this class of differential forms,

$$\mathsf{d}^k \mathcal{P}\Lambda^k(\mathcal{T}) \subseteq \mathcal{P}\Lambda^{k+1}(\mathcal{T}). \tag{49}$$

and in particular, we have a differential complex

$$\dots \xrightarrow{\mathsf{d}^{k-1}} \mathcal{P}\Lambda^k(\mathcal{T}) \xrightarrow{\mathsf{d}^k} \mathcal{P}\Lambda^{k+1}(\mathcal{T}) \xrightarrow{\mathsf{d}^{k+1}} \dots \tag{50}$$

Having associated an admissible sequence type $\mathcal{P}_T$ to each $T \in \mathcal{T}$, we say that the *hierarchy condition holds* if for all $F, T \in \mathcal{T}$ we have

$$F \in \Delta(T) \implies \mathcal{P}_F \leq \mathcal{P}_T. \tag{51}$$

We assume the hierarchy condition throughout this section. In order to simplify the notation, we will write $\mathcal{P}\Lambda^k(T) := \mathcal{P}_T\Lambda^k(T)$ from here on.

Remark 5.1.
The general idea of the hierarchy condition is that the polynomial order associated to a simplex is at least the polynomial order associated to any subsimplex. Imposing such a condition is common in literature on *hp* finite element methods (22). Indeed, if $(\mathcal{P}_T)_{T\in\mathcal{T}}$ violates the hierarchy condition, then there exists a family of sequence types $(\mathcal{S}_T)_{T\in\mathcal{T}}$ that satisfies the hierarchy condition and yields the same space $\mathcal{P}\Lambda^k(\mathcal{T})$. This is analogous to what is called *minimum rule* by (23)

The geometric decomposition of finite element spaces is a concept of paramount importance. To have geometric decompositions at our disposal, we make the additional assumption that we are given extension operators between finite element spaces over simplices. Specifically, we assume to have linear *local extension operators*

$$\operatorname{ext}^k_{F,T} : \mathring{\underline{\mathcal{P}}}\Lambda^k(F) \to \mathcal{P}\Lambda^k(T) \tag{52}$$

for every pair $F \in \Delta(T)$ with $T \in \mathcal{T}$, such that

$$\operatorname{ext}^k_{F,F} \omega = \omega, \quad \omega \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F) \tag{53a}$$

for all $F \in \mathcal{T}$, such that

$$\operatorname{tr}^k_{T,F} \operatorname{ext}^k_{f,T} = \operatorname{ext}^k_{f,F} \tag{53b}$$

for all $T \in \mathcal{T}$ with $F \in \Delta(T)$ and $f \in \Delta(F)$, and such that

$$\operatorname{tr}^k_{T,G} \operatorname{ext}^k_{F,T} = 0, \tag{53c}$$

for all $T \in \mathcal{T}$ with $F, G \in \Delta(T)$ but $F \notin \Delta(G)$.

For each $F \in \mathcal{T}$ we then define the associated *global extension operator*,

$$\mathrm{Ext}^k_F : \mathring{\underline{\mathcal{P}}}\Lambda^k(F) \to C^\infty\Lambda^k(\mathcal{T}), \quad \mathring{\omega} \mapsto \bigoplus_{\substack{T\in\mathcal{T}\\ F\in\Delta(T)}} \mathrm{ext}^k_{F,T}\,\mathring{\omega}. \tag{54}$$

It follows from (53b) that this mapping indeed takes values in $C^\infty\Lambda^k(\mathcal{T})$. It is clear from definitions, moreover, that

$$\mathrm{Ext}^k_F\left(\mathring{\underline{\mathcal{P}}}\Lambda^k(F)\right) \subseteq \mathcal{P}\Lambda^k(\mathcal{T}). \tag{55}$$

We note that $\mathrm{Ext}^k_F\,\omega$ for $\omega \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F)$ vanishes on all simplices of $\mathcal{T}$ that do not contain F as a subsimplex.

Example 5.2.
Extension operators $\mathrm{ext}^k_{F,T}$ with these properties are constructed by (5) with a case distinction depending on whether $\mathring{\underline{\mathcal{P}}}\Lambda^k(F) = \mathring{\underline{\mathcal{P}}}_r\Lambda^k(F)$ or $\mathring{\underline{\mathcal{P}}}\Lambda^k(F) = \mathring{\underline{\mathcal{P}}}^-_r\Lambda^k(F)$ for some $r \in \mathbb{N}$. We give a brief outline.

Let $T \in \mathcal{T}^n$ and $F \in \Delta(T)^m$, and let $\{v^T_0,\dots,v^T_n\}$ and $\{v^F_0,\dots,v^F_m\}$ be the respective set of vertices. For $\alpha \in A(m)$ we let $\alpha_{F,T} \in A(n)$ be uniquely defined by $\alpha_{F,T}(j) = \alpha(i)$ if $v^T_j = v^F_i$ and $\alpha_{F,T}(j) = 0$ otherwise for $j \in \{0,\dots,n\}$ and $i \in \{0,\dots,m\}$. For $\sigma \in \Sigma(a:k,m)$ we let $\sigma_{F,T} \in \Sigma(a:k,n)$ be uniquely defined by $v^T_{\sigma_{F,T}} = v^F_{\sigma(i)}$ for $a \le i \le k$.

Now, on the one hand, there exists a well-defined linear operator

$$\mathrm{ext}^{r,k,-}_{F,T} : \mathcal{P}^-_r\Lambda^k(F) \to \mathcal{P}^-_r\Lambda^k(T)$$

which is uniquely defined by

$$\mathrm{ext}^{r,k,-}_{F,T}\,\lambda^\alpha_F\phi^F_\rho = \lambda^{\alpha_{F,T}}_T\,\phi^T_{\rho_{F,T}}, \quad \alpha \in A(r-1,m), \quad \rho \in \Sigma(0:k,0:m).$$

The restriction of $\mathrm{ext}^{r,k,-}_{F,T}$ to $\mathring{\underline{\mathcal{P}}}^-_r\Lambda^k(F)$ provides the required mapping. On the other hand, there exists a well-defined linear operator

$$\mathrm{ext}^{r,k}_{F,T} : \mathcal{P}_r\Lambda^k(F) \to \mathcal{P}_r\Lambda^k(T)$$

which is uniquely defined by

$$\mathrm{ext}^{r,k}_{F,T}\,\lambda^\alpha_F \mathsf{d}\lambda^F_\sigma = \lambda^{\alpha_{F,T}}_T\,\Psi^{\alpha,F,T}_{\sigma_{F,T}}, \quad \alpha \in A(r,m), \quad \sigma \in \Sigma(1:k,0:m),$$

where we have used

$$\Psi^{\alpha,F,T}_{\sigma_{F,T}} := \Psi^{\alpha,F,T}_{\sigma_{F,T}(1)} \wedge \dots \wedge \Psi^{\alpha,F,T}_{\sigma_{F,T}(k)},$$

$$\Psi^{\alpha,F,T}_i := \mathsf{d}\lambda^T_i - \frac{\alpha_{F,T}(i)}{r}\sum_{j=0}^m \mathsf{d}\lambda^T_{\imath_{F,T}(j)}, \quad 0 \le i \le n.$$

The restriction of $\mathrm{ext}^{r,k}_{F,T}$ to $\mathring{\underline{\mathcal{P}}}_r\Lambda^k(F)$ provides the required mapping.

With the local extension operators we can describe the geometric decomposition of $\mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$. The hierarchy condition is crucial for this endeavor.

Consider $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$. We define $\omega^{\mathcal{W}} \in \mathcal{P}\Lambda^k(\mathcal{T})$ by

$$\omega^{\mathcal{W}} := \sum_{F\in\mathcal{T}^k} \mathrm{vol}(F)^{-1}\left(\int_F \mathrm{tr}^k_F\,\omega\right)\mathrm{Ext}^k_F\,\mathrm{vol}_F\,. \tag{56}$$

We then define recursively for every $m \in \{k, \dots, n\}$

$$\mathring{\omega}_F := \operatorname{tr}^k_F \left(\omega - \omega^{\mathcal{W}} - \sum_{l=k}^{m-1} \omega^l \right), \quad F \in \mathcal{T}^m, \tag{57}$$

$$\omega^m := \sum_{F \in \mathcal{T}^m} \operatorname{Ext}^k_F \mathring{\omega}_F. \tag{58}$$

The following theorem shows that these definitions are well-defined and give a decomposition of ω.

Theorem 5.3.
Let $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$. Then we have $\mathring{\omega}_F \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F)$ for every $F \in \mathcal{T}$ and

$$\omega = \omega^{\mathcal{W}} + \sum_{m=k}^{n} \omega^m. \tag{59}$$

Proof. By construction of $\omega^{\mathcal{W}}$ we have

$$\int_F \operatorname{tr}^k_F \omega^{\mathcal{W}} = \int_F \operatorname{tr}^k_F \omega, \quad F \in \mathcal{T}^k.$$

By definition, $\operatorname{tr}^k_F \left(\omega - \omega^{\mathcal{W}} \right) \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F)$ for every $F \in \mathcal{T}^k$. With ω^k as defined above, we see

$$\operatorname{tr}^k_F \left(\omega - \omega^{\mathcal{W}} - \omega^k \right) = 0, \quad F \in \mathcal{T}^k.$$

Let us now suppose that for some $m \in \{k, \dots, n-1\}$ we have shown

$$\operatorname{tr}^k_f \left(\omega - \omega^{\mathcal{W}} - \sum_{l=k}^{m} \omega^l \right) = 0, \quad f \in \mathcal{T}^m.$$

By definition we have $\mathring{\underline{\mathcal{P}}}\Lambda^k(F) = \mathring{\mathcal{P}}\Lambda^k(F)$ for $F \in \mathcal{T}^{m+1}$, and $\mathring{\omega}_F \in \mathring{\mathcal{P}}\Lambda^k(F)$ for $F \in \mathcal{T}^{m+1}$. We conclude that ω^{m+1} is well-defined and that

$$\operatorname{tr}^k_F \left(\omega - \omega^{\mathcal{W}} - \sum_{l=k}^{m+1} \omega^l \right) = 0, \quad F \in \mathcal{T}^{m+1}.$$

An induction argument then provides (59). The proof is complete. □

Lemma 5.4.
Let $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ and $F \in \mathcal{T}$. Then we have $\omega_F = 0$ if and only if

$$\operatorname{tr}^k_f \omega^{\mathcal{W}} = 0, \quad f \in \Delta(F),$$
$$\mathring{\omega}_f = 0, \quad f \in \Delta(F)^k.$$

Proof. For any $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ and $F \in \mathcal{T}^m$ we observe

$$\begin{aligned} \omega_F &= \operatorname{tr}^k_F \omega^{\mathcal{W}} + \sum_{k \leq m \leq n} \sum_{f \in \mathcal{T}^m} \operatorname{tr}^k_F \operatorname{Ext}^k_{f,\mathcal{T}} \mathring{\omega}_f \\ &= \sum_{f \in \Delta(F)^k} \operatorname{vol}(F)^{-1} \left(\int_f \operatorname{tr}^k_f \omega \right) \operatorname{Ext}^k_{f,F} \operatorname{vol}_F + \sum_{f \in \Delta(F)} \operatorname{Ext}^k_{f,F} \mathring{\omega}_f. \end{aligned}$$

If $k = m$, then $\omega_F = \operatorname{tr}^k_F \omega^{\mathcal{W}} + \mathring{\omega}_F$, and the claims follows by this being a direct sum. If $k < m$, let us assume that the claim holds true for all $f \in \mathcal{T}$ with $k \leq \dim f < m$.

Then $\omega_F = \mathring{\omega}_F$, which again proves the claim. The lemma now follows from an induction argument. □

Lemma 5.5.
For $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ we have $\omega \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$ if and only if

$$\begin{aligned} \mathring{\omega}_F &= 0, \quad F \in \mathcal{U}, \\ \omega_F^{\mathcal{W}} &= 0, \quad F \in \mathcal{U}^k. \end{aligned}$$

Proof. This is a simple consequence of Lemma 5.4. □

Lemma 5.6.
For $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ we have $\omega = 0$ if and only if

$$\begin{aligned} \mathring{\omega}_F &= 0, \quad F \in \mathcal{T}, \\ \omega_F^{\mathcal{W}} &= 0, \quad F \in \mathcal{T}^k. \end{aligned}$$

Proof. This follows from Lemma 5.5 applied to the case $\mathcal{U} = \mathcal{T}$. □

Theorem 5.7.
We have

$$\mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) = \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}) \oplus \bigoplus_{F \in \mathcal{T}\setminus\mathcal{U}} \operatorname{Ext}_F^k \mathring{\underline{\mathcal{P}}}\Lambda^k(F).$$

A modification of the geometric decomposition will be helpful to us in the sequel.

Lemma 5.8.
Let $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$. Then there exist unique $\mathring{\omega}_F \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F)$ for $F \in \mathcal{T}$ such that

$$\omega = I_{\mathcal{W}}^k \omega + \sum_{k \leq m \leq n} \sum_{F \in \mathcal{T}^m} \operatorname{Ext}_F^k \mathring{\omega}_F^m. \tag{60}$$

Proof. Let $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$. The trace of $I_{\mathcal{W}}^k\omega - \omega$ over any simplex $F \in \mathcal{T}^k$ has vanishing integral. The claim follows from applying Theorem 5.3 to $I_{\mathcal{W}}^k - \omega$. □

In the remainder of this section we define the canonical finite element interpolant and study some of its properties. The basic ideas have already been used in priori literature (23), but we apply some modifications and extensions. Our construction explicitly calculates the geometric decomposition of the interpolating differential form. We first define

$$J_{\mathcal{W}}^k : C^\infty\Lambda^k(\mathcal{T}) \to \mathcal{P}\Lambda^k(\mathcal{T}), \quad \omega \mapsto \sum_{F \in \mathcal{T}^k} \operatorname{vol}(F)^{-1} \left(\int_F \omega \right) \cdot \operatorname{Ext}_F^k \operatorname{vol}_F. \tag{61}$$

Subsequently for $m \in \{k, \dots, n\}$ we define recursively

$$J_m^k : C^\infty\Lambda^k(\mathcal{T}) \to \mathcal{P}\Lambda^k(\mathcal{T}), \quad \omega \mapsto \sum_{F \in \mathcal{T}^m} \operatorname{Ext}_F^k J_F^k \omega, \tag{62}$$

where for each $F \in \mathcal{T}^m$ we define

$$J_F^k : C^\infty\Lambda^k(\mathcal{T}) \to \mathring{\underline{\mathcal{P}}}\Lambda^k(F) \tag{63}$$

by requiring $J^k_F\omega$ for $\omega \in C^\infty\Lambda^k(\mathcal{T})$ to be the unique solution of

(64a)
$$\int_F \left\langle J^k_F\omega, \mathsf{d}^{k-1}\rho \right\rangle = \int_F \left\langle \operatorname{tr}^k_F \left(\omega - J^k_{\mathcal{W}}\omega - \sum_{k=l}^{m-1} J^k_l \omega \right), \mathsf{d}^{k-1}\rho \right\rangle, \quad \rho \in \mathring{\underline{\mathcal{P}}}\Lambda^{k-1}(F),$$
(64b)
$$\int_F \left\langle \mathsf{d}^k J^m_F\omega, \mathsf{d}^k\beta \right\rangle = \int_F \left\langle \mathsf{d}^k \operatorname{tr}^k_F \left(\omega - J^k_{\mathcal{W}}\omega - \sum_{k=l}^{m-1} J^k_l \omega \right), \mathsf{d}^k\beta \right\rangle, \quad \beta \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F).$$

That $J^k_F\omega$ is well-defined follows easily from Lemma 3.3. We then set

$$I^k_{\mathcal{P}} : C^\infty\Lambda^k(\mathcal{T}) \to \mathcal{P}\Lambda^k(\mathcal{T}), \quad \omega \mapsto J^k_{\mathcal{W}}\omega + J^k_k\omega + \cdots + J^k_n\omega. \tag{65}$$

We show that the operator $I^k_{\mathcal{P}}$ acts as the identity on $\mathcal{P}\Lambda^k(\mathcal{T})$, and its constituents J^k_F reproduce the geometric decomposition.

Lemma 5.9.
For each $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ we have $I^k_{\mathcal{P}}\omega = \omega$. Moreover $J^k_{\mathcal{W}}\omega = \omega^{\mathcal{W}}$ and $J^k_F\omega = \mathring{\omega}_F$ for each $F \in \mathcal{T}$.

Proof. Let $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$. We have $J^k_{\mathcal{W}}\omega = \omega^{\mathcal{W}}$ by definition. For $F \in \mathcal{T}^k$ we find $\operatorname{tr}^k_F \left(\omega - \omega^{\mathcal{W}}\right) \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F)$, and $J^k_F\omega = \mathring{\omega}_F$ follows easily. Next, let $m \in \{k, \dots, n-1\}$ and suppose that $J^k_F\omega = \mathring{\omega}_F$ for $F \in \mathcal{T}$ with $\dim F \leq m$. Let $F \in \mathcal{T}^{m+1}$. From definitions we conclude that

$$\operatorname{tr}^k_F \left(\omega - \omega^{\mathcal{W}} - \sum_{l=k}^{m-1} J^l\omega \right) \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F).$$

It follows that $J^k_F\omega = \mathring{\omega}_F$ and hence $J^k_m\omega = \omega^m$. An induction argument completes the proof. □

Lemma 5.10.
Let $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$. If

$$\int_F \operatorname{tr}^k_F \omega' = 0, \quad F \in \mathcal{T}^k, \tag{66a}$$

$$\int_F \left\langle \operatorname{tr}^k_F \omega', \mathsf{d}^{k-1}\rho \right\rangle_g = 0, \quad \rho \in \mathring{\underline{\mathcal{P}}}\Lambda^{k-1}(F), \quad F \in \mathcal{T}, \tag{66b}$$

$$\int_F \left\langle \mathsf{d}^k \operatorname{tr}^k_F \omega', \mathsf{d}^k\beta \right\rangle_g = 0, \quad \beta \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F) \quad F \in \mathcal{T}, \tag{66c}$$

then $\omega = 0$.

Proof. This follows from (5.9) and an induction argument. □

An auxiliary results yields an alternative characterization of $I^k_{\mathcal{P}}$.

Lemma 5.11.
Let $\omega \in C^\infty\Lambda^k(\mathcal{T})$ and $\omega' \in \mathcal{P}\Lambda^k(\mathcal{T})$. We have $\omega' = I^k_{\mathcal{P}}\omega$ if and only if

$$\int_F \operatorname{tr}^k_F \omega' = \int_F \operatorname{tr}^k_F \omega, \quad F \in \mathcal{T}^k, \tag{67a}$$

$$\int_F \left\langle \operatorname{tr}^k_F \omega', \mathsf{d}^{k-1}\rho \right\rangle = \int_F \left\langle \operatorname{tr}^k_F \omega, \mathsf{d}^{k-1}\rho \right\rangle, \quad \rho \in \mathring{\underline{\mathcal{P}}}\Lambda^{k-1}(F), \quad F \in \mathcal{T}, \tag{67b}$$

$$\int_F \left\langle \mathsf{d}^k \operatorname{tr}^k_F \omega', \mathsf{d}^k\beta \right\rangle = \int_F \left\langle \mathsf{d}^k \operatorname{tr}^k_F \omega, \mathsf{d}^k\beta \right\rangle, \quad \beta \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F) \quad F \in \mathcal{T}. \tag{67c}$$

Proof. Let $\omega \in C^\infty\Lambda^k(\mathcal{T})$. We verify that $I^k_{\mathcal{P}}\omega$ satisfies (67) by rearranging the terms in (63) and the assumptions on the extension operators. If $\omega' \in \mathcal{P}\Lambda^k(\mathcal{T})$ is another solution to (67), then we obtain $\omega' = I^k_{\mathcal{P}}\omega$ by applying Lemma 5.10 to $\omega' - I^k_{\mathcal{P}}\omega$. The claim follows by an induction argument. □

Lemma 5.12.
Let $\omega \in C^\infty\Lambda^k(\mathcal{T})$ and $F \in \mathcal{T}$. If $\omega_F = 0$ then $\operatorname{tr}^k_F \left(I^k_{\mathcal{P}}\omega\right) = 0$.

Proof. Unfolding definitions we find

$$\begin{aligned}
\operatorname{tr}^k_F \left(I^k_{\mathcal{P}}\omega\right) &= \operatorname{tr}^k_F J^k_{\mathcal{W}}\omega + \sum_{m=k}^{n} \sum_{f \in \mathcal{T}^m} \operatorname{tr}^k_F \operatorname{Ext}^k_{f,\mathcal{T}} J^k_f \omega \\
&= \sum_{f \in \Delta(F)^k} \operatorname{vol}(F)^{-1} \left(\int_f \operatorname{tr}^k_f \omega \right) \operatorname{Ext}^k_{f,F} \operatorname{vol}_F + \sum_{f \in \Delta(F)} \operatorname{Ext}^k_{f,F} J^k_f \omega.
\end{aligned}$$

If $\dim F = k$, then the claim follows from the direct sum decomposition (22). If $\dim F > k$, suppose that the claim has been proven for $f \in \Delta(F)$. Since $\omega_F = 0$ we have $\omega_f = 0$ for $f \in \Delta(F)$. Hence $\operatorname{tr}^k_F \left(I^k_{\mathcal{P}}\omega\right) = J^k_F\omega$, from which $\operatorname{tr}^k_F \left(I^k_{\mathcal{P}}\omega\right) = 0$ follows. An induction argument proves the claim. □

Lemma 5.13.
If $\omega \in C^\infty\Lambda^k(\mathcal{T},\mathcal{U})$, then $I^k_{\mathcal{P}}\omega \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$.

Proof. This is an immediate consequence of Lemma 5.12 above. □

It remains to show that the canonical interpolant commutes with the exterior derivative, so we have a commuting diagram

$$\begin{array}{ccccccc}
\dots \xrightarrow{\mathsf{d}^{k-1}} & \mathcal{C}^\infty\Lambda^k(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^k} & \mathcal{C}^\infty\Lambda^{k+1}(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^{k+1}} \dots \\
& \Big\downarrow {\scriptstyle I^k_{\mathcal{P}}} & & \Big\downarrow {\scriptstyle I^{k+1}_{\mathcal{P}}} & \\
\dots \xrightarrow{\mathsf{d}^{k-1}} & \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^k} & \mathcal{P}\Lambda^{k+1}(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^{k+1}} \dots
\end{array}$$

This the subject of the following lemma.

Lemma 5.14.
We have $\mathsf{d}^k I^k_{\mathcal{P}}\omega = I^{k+1}_{\mathcal{P}}\mathsf{d}^k\omega$ for $\omega \in C^\infty\Lambda^k(\mathcal{T})$.

Proof. Let $\omega \in C^\infty\Lambda^k(\mathcal{T},\mathcal{U})$. For $F \in \mathcal{T}^{k+1}$ we observe

$$\begin{aligned}\int_F \operatorname{tr}_F^{k+1} \mathsf{d}^k I_{\mathcal{P}}^k \omega &= \int_F \operatorname{tr}_F^{k+1} \mathsf{d}^k J_{\mathcal{W}}^k \omega = \int_F \mathsf{d}^k \operatorname{tr}_F^k J_{\mathcal{W}}^k \omega \\ &= \int_{\partial F} \operatorname{tr}_F^k J_{\mathcal{W}}^k \omega = \int_{\partial F} \operatorname{tr}_F^k \omega \\ &= \int_F \mathsf{d}^k \operatorname{tr}_F^k \omega = \int_F \operatorname{tr}_F^{k+1} \mathsf{d}^k \omega = \int_F \operatorname{tr}_F^{k+1} J_{\mathcal{W}}^{k+1} \mathsf{d}^k \omega = \int_F \operatorname{tr}_F^{k+1} I_{\mathcal{P}}^{k+1} \mathsf{d}^k \omega.\end{aligned}$$

Let $F \in \mathcal{T}^m$ with $k \leq m \leq n$. For $\rho \in \mathring{\mathcal{P}}\Lambda^k(F)$ we find

$$\begin{aligned}\int_F \left\langle I_{\mathcal{P}}^{k+1} \mathsf{d}^k \omega, \mathsf{d}^k \rho \right\rangle &= \int_F \left\langle \mathsf{d}^k \omega, \mathsf{d}^k \rho \right\rangle \\ &= \int_F \left\langle \mathsf{d}^k I_{\mathcal{P}}^k \omega, \mathsf{d}^k \rho \right\rangle = \int_F \left\langle \mathsf{d}^k I_{\mathcal{P}}^k \omega, \mathsf{d}^k \rho \right\rangle.\end{aligned}$$

For $\beta \in \mathring{\mathcal{P}}\Lambda^{k+1}(F)$ we find

$$\begin{aligned}\int_F \left\langle \mathsf{d}^{k+1} I_{\mathcal{P}}^{k+1} \mathsf{d}^k \omega, \mathsf{d}^{k+1} \beta \right\rangle &= \int_F \left\langle \mathsf{d}^{k+1} \mathsf{d}^k \omega, \mathsf{d}^{k+1} \beta \right\rangle \\ &= \int_F \left\langle \mathsf{d}^{k+1} \mathsf{d}^k I_{\mathcal{P}}^k \omega, \mathsf{d}^{k+1} \beta \right\rangle = 0.\end{aligned}$$

In conjunction with Lemma 5.11, the desired result follows. □

Remark 5.15.
In the definition of the commuting interpolant and in Lemma 5.11 we have implicitly used degrees of freedom associated with simplices of the triangulation. Our formulation of the degrees of freedom, however, uses an arbitrary Riemannian metric. When we restrict to finite element de Rham complexes of spaces of *uniform* polynomial order, then the degrees of freedom have canonical representations not involving a Riemannian metric.

6. Partially Localized Flux Reconstruction

In this section we approach the main result of this article. Our investigations on the structure of finite element spaces allow us to formalize a partially localized method of flux reconstruction. The subject of flux reconstruction is to solve the first-order differential equation $\mathsf{d}^{k-1}\xi = \omega$, where $\omega \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$ is the data and $\xi \in \mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U})$ is the unknown. Assuming that a solution exists, we wish to efficiently compute one of the possible solutions. Problems of this type appear in a posteriori error estimation.

The problem of flux reconstruction amounts to determining a generalized inverse of the operator $\mathsf{d}^{k-1} : \mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U}) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$. In this article we contribute a method to reduce this problem to the lowest-order case. It only remains to find a generalized inverse of $\mathsf{d}^{k-1} : \mathcal{W}\Lambda^{k-1}(\mathcal{T},\mathcal{U}) \to \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U})$. The higher order aspects of the problem are treated in local problems associated to simplices which can be solved independently form each other. This is a fundamental result on the structure of higher order finite element spaces that is not only of theoretical appeal, but also relevant in numerical algorithms.

Before we formulate the main result we introduce several generalized inverses. First we fix a generalized inverse of the exterior derivative between Whitney forms. Specifically, we assume that we have a linear mapping

$$\mathsf{P}^k_{\mathcal{W}} : \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}) \to \mathcal{W}\Lambda^{k-1}(\mathcal{T},\mathcal{U}) \tag{68}$$

such that

$$\mathsf{d}^{k-1}\mathsf{P}^k_{\mathcal{W}}\mathsf{d}^{k-1}\xi = \mathsf{d}^{k-1}\xi, \quad \xi \in \mathcal{W}\Lambda^{k-1}(\mathcal{T},\mathcal{U}). \tag{69}$$

In particular, $\omega = \mathsf{d}^{k-1}\mathsf{P}^k_{\mathcal{W}}\omega$ whenever $\omega \in \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U})$ is the exterior derivative of a Whitney form in $\mathcal{W}\Lambda^{k-1}(\mathcal{T},\mathcal{U})$. Similarly, for each simplex $F \in \mathcal{T}$ we fix a generalized derivative

$$\mathsf{P}^k_F : \mathring{\underline{\mathcal{P}}}\Lambda^k(F) \to \mathring{\underline{\mathcal{P}}}\Lambda^{k-1}(F) \tag{70}$$

such that

$$\mathsf{d}^{k-1}\mathsf{P}^k_F\mathsf{d}^{k-1}\xi = \mathsf{d}^{k-1}\xi, \quad \xi \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F). \tag{71}$$

We have $\omega = \mathsf{d}^{k-1}\mathsf{P}^k_F\omega$ whenever $\omega \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F)$ is the exterior derivative of a Whitney form in $\mathring{\underline{\mathcal{P}}}\Lambda^{k-1}(F)$. The existence of a mapping $\mathsf{P}^k_{\mathcal{W}}$ and mappings P^k_F with such properties is elementary.

Remark 6.1.
There is no canonical choice in fixing the generalized inverses. Upon fixing a Hilbert space structure on the Whitney forms, however, a natural choice is the Moore-Penrose pseudoinverse of $\mathsf{d}^{k-1} : \mathcal{W}\Lambda^{k-1}(\mathcal{T},\mathcal{U}) \to \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U})$. This Moore-Penrose pseudoinverse provides the least-squares solution of the problem. Entirely analogous statements hold for choosing the generalizes inverses P^k_F.

Assuming to have fixed generalized inverses as above, we provide the partially localized flux reconstruction without further ado.

Theorem 6.2.
Suppose that $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ with $\mathsf{d}^k\omega = 0$. For $m \in \{k,\dots,n\}$ we let

$$\xi^m := \sum_{F\in\mathcal{T}^m} \mathrm{Ext}^{k-1}_F\, \mathsf{P}^k_F\, \mathrm{tr}^k_F \left(\omega - I^k_{\mathcal{W}}\omega - \sum_{l=k}^{m-1} \mathsf{d}^{k-1}\xi^l \right). \tag{72}$$

Then

$$I^k_{\mathcal{W}}\omega + \mathsf{d}^{k-1}\left(\sum_{m=k}^{n} \xi^m \right) = \omega. \tag{73}$$

If there exists $\xi \in \mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U})$ with $\mathsf{d}^{k-1}\xi = \omega$, then

$$\mathsf{d}^{k-1}\left(\mathsf{P}^k_{\mathcal{W}} I^k_{\mathcal{W}}\omega + \sum_{m=k}^{n} \xi^m \right) = \omega. \tag{74}$$

Proof. We use the modified geometric decomposition (Lemma 5.8) to write

$$\omega = I^k_{\mathcal{W}}\omega + \sum_{m=k}^{n} \sum_{F\in\mathcal{T}^m} \mathrm{Ext}^k_F\, \mathring{\omega}_F,$$

where $\mathring{\omega}_F \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F)$ for each $F \in \mathcal{T}$. We thus find for $F \in \mathcal{T}^k$ that

$$\operatorname{tr}_F^k \left(\omega - I_{\mathcal{W}}^k \omega \right) \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F).$$

The proof will be completed by an induction argument. For each $F \in \mathcal{T}$ we set

$$\theta_F := \operatorname{tr}_F^k \left(\omega - I_{\mathcal{W}}^k \omega - \sum_{l=k}^{\dim F - 1} \mathsf{d}^{k-1} \xi^l \right).$$

Let $m \in \{k, \dots, n-1\}$. Suppose that $\theta_f \in \mathring{\underline{\mathcal{P}}}\Lambda^k(f)$ for each $f \in \mathcal{T}^m$, which is certainly true if $m = k$. Then ξ^m as in (72) is well-defined. By assumptions on ω we find

$$\begin{aligned} \mathsf{d}^k \theta_f &= \mathsf{d}^k \operatorname{tr}_f^k \left(\omega - I_{\mathcal{W}}^k \omega - \sum_{l=k}^{m-1} \mathsf{d}^{k-1} \xi^l \right) \\ &= \operatorname{tr}_f^k \left(\mathsf{d}^k \omega - \mathsf{d}^k I_{\mathcal{W}}^k \omega - \mathsf{d}^k \sum_{l=k}^{m-1} \mathsf{d}^{k-1} \xi^l \right) \\ &= \operatorname{tr}_f^k \left(\mathsf{d}^k \omega - I_{\mathcal{W}}^{k+1} \mathsf{d}^k \omega \right) = 0, \end{aligned}$$

and conclude that $\mathsf{d}^{k-1} P_f^k \theta_f = \theta_f$. In particular,

$$\operatorname{tr}_f^k \mathsf{d}^{k-1} \xi^m = \mathsf{d}^{k-1} P_f \theta_f = \operatorname{tr}_f^k \left(\omega - I_{\mathcal{W}}^k \omega - \sum_{l=k}^{m-1} \mathsf{d}^{k-1} \xi^l \right). \tag{75}$$

If $m < n$, then $\theta_F \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F)$ for each $F \in \mathcal{T}^{m+1}$. The argument may be iterated until $m = n$. In the latter case (75) provides (73).

Finally, if there exists $\xi \in \mathcal{P}\Lambda^{k-1}(\mathcal{T}, \mathcal{U})$ with $\mathsf{d}^{k-1}\xi = \omega$, then

$$I_{\mathcal{W}}^k \omega = I_{\mathcal{W}}^k \mathsf{d}^{k-1} \xi = \mathsf{d}^{k-1} I_{\mathcal{W}}^{k-1} \xi.$$

and hence $\mathsf{d}^{k-1} \mathsf{P}_{\mathcal{W}}^k I_{\mathcal{W}}^k \xi = I_{\mathcal{W}}^{k-1} \xi$, which shows (74). This completes the proof. □

The theorem states that for every $\omega \in \mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{U})$ with $\mathsf{d}^k \omega = 0$ there exists $\xi^{hi} \in \mathcal{P}\Lambda^{k-1}(\mathcal{T}, \mathcal{U})$ such that $\omega = I_{\mathcal{W}}^k \omega + \mathsf{d}^k \xi^{hi}$. If additionally ω is the exterior derivative of a member of $\mathcal{P}\Lambda^{k-1}(\mathcal{T}, \mathcal{U})$, then there exists $\xi^{lo} \in \mathcal{W}\Lambda^{k-1}(\mathcal{T}, \mathcal{U})$ with $\mathsf{d}^{k-1} \xi^{lo} = I_{\mathcal{W}}^k \omega$. Thus $\xi := \xi^{lo} + \xi^{hi}$ is a solution of $\mathsf{d}^{k-1} \xi = \omega$.

As a simple first application we address the dimension of the cohomology classes of the finite element de Rham complex. This is a new proof of a result which has been shown before (6; 19; 31) with different techniques. Conceptually, this shows us the cohomological information are encoded completely in the lowest order component of the finite element de Rham complex.

Lemma 6.3.
The commuting interpolator $I_{\mathcal{W}}^k : \mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{U}) \to \mathcal{W}\Lambda^k(\mathcal{T}, \mathcal{U})$ induces isomorphisms on cohomology.

Proof. Let $\omega \in \mathcal{W}\Lambda^k(\mathcal{T}, \mathcal{U})$ with $\mathsf{d}^k \omega = 0$. If $\omega \notin \mathsf{d}^{k-1} \mathcal{W}\Lambda^{k-1}(\mathcal{T}, \mathcal{U})$, then $\omega \notin \mathsf{d}^{k-1} \mathcal{P}\Lambda^{k-1}(\mathcal{T}, \mathcal{U})$, since the canonical interpolant commutes with the exterior derivative. Hence $I_{\mathcal{W}}^k$ induces a surjection on cohomology. Conversely, suppose that $\omega \in \mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{U})$ and $\omega \notin \mathsf{d}^{k-1} \mathcal{P}\Lambda^{k-1}(\mathcal{T}, \mathcal{U})$. There exists $\xi \in \mathcal{P}\Lambda^{k-1}(\mathcal{T}, \mathcal{U})$ such that $\omega = \mathsf{d}^{k-1}\xi + I_{\mathcal{W}}^k \omega$. Now $\omega \notin \mathsf{d}^{k-1} \mathcal{P}\Lambda^{k-1}(\mathcal{T}, \mathcal{U})$ implies $I_{\mathcal{W}}^k \omega \notin \mathsf{d}^{k-1} \mathcal{W}\Lambda^{k-1}(\mathcal{T}, \mathcal{U})$. Hence $I_{\mathcal{W}}^k$ is injective on cohomology. This completes the proof. □

The partially localized flux reconstruction is relevant from a computational point of view too. In order to compute a solution of $\mathsf{d}^{k-1}\xi = \omega$ for given $\omega \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$ we treat this first-order equation as a least-squares problem. This means that we fix a Hilbert space structure on the finite element spaces and compute the action of the Moore-Penrose pseudoinverse of $\mathsf{d}^{k-1} : \mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U}) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$. This is a standard topic of numerical linear algebra, but the spectral properties of the operator $\mathsf{d}^{k-1} : \mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U}) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$ for higher polynomial order can be disadvantageous. The condition number of the least-squares problem grows algebraically with the polynomial degree, which negatively affects the performance of the numerical methods. The complexity of the problem on a higher order spaces is comparable to computing the flux variable in a mixed finite element method.

But Theorem 6.2 shows us how to avoid solving a global problem on a high order finite element space. As outline above, with a block of local mutually independent computations we split the main problem into two independent subproblems: one subproblem involving Whitney forms and another subproblem involving higher order contributions. In the former subproblem we seek a flux reconstruction $\xi^{lo} \in \mathcal{W}\Lambda^{k-1}(\mathcal{T},\mathcal{U})$ for $I^k_{\mathcal{P}}\omega \in \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U})$. Hence we still need to solve a global least-squares problem, but this time only for the operator $\mathsf{d}^{k-1} : \mathcal{W}\Lambda^{k-1}(\mathcal{T},\mathcal{U}) \to \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U})$ over finite element spaces of lowest order. In the second subproblem we calculate ξ^{hi} by iterating over the dimension of the simplices in $\mathcal{T}$ from lowest to highest; at each step we solve an block of mutually independent local subproblems is solved. In particular, at each step the computation is amenable to parallelization.

In this sense the flux reconstruction is partially localized: the only remaining global operation involves a finite element space of merely lowest order instead of the full finite element space. A fully localized flux reconstruction is feasible when additional structure is provided; this will be crucial to our application in the next section.

Remark 6.4.
Instead of solving a sequence of parallelizable blocks of local mutually independent computations, we can rearrange the computations such that, at the cost of redundant computations, we need process only one parallelizable block of mutually independent local problems associated to full-dimensional simplices.

Remark 6.5.
The L^2 stability of the global lowest-order problem depends only on the mesh quality and the domain, and the L^2 stability of the local problems depends only on the mesh quality and the polynomial order. Whether the dependency on the polynomial order can be dropped remains for future research (but see 13).

Remark 6.6.
A flux reconstruction for discrete distributional differential forms is known (18). The construction in that reference allows to estimate the constants in discrete Poincaré-Friedrichs inequalities, but is not applicable as such to generalize the Braess-Schöberl error estimator, as we will do in the next section.

Example 6.7.
We illustrate the partially localized flux reconstruction with two-dimensional finite

element de Rham complexes. Assume that $\Omega \subset \mathbb{R}^2$ is a bounded simply-connected Lipschitz domain, that $\mathcal{T}$ be a triangulation of Ω, and that $\mathcal{U} \subset \mathcal{T}$ triangulates $\partial\Omega$.

We let $\mathcal{P}_r(\mathcal{T},\mathcal{U})$ denote the functions over Ω that are piecewise polynomial of order r with respect to $\mathcal{T}$ and satisfy Dirichlet boundary conditions. We let $\mathcal{P}_{r,\mathsf{DC}}(\mathcal{T})$ be functions the functions over Ω that are piecewise polynomial of order r with respect to $\mathcal{T}$. We let $\mathcal{P}_r(\mathcal{T},\mathcal{U}) \subseteq \mathcal{P}_{r,\mathsf{DC}}(\mathcal{T})$ be the subspace whose members have a square-integrable weak gradient and satisfy Dirichlet boundary conditions. Finally, we let $\mathbf{RT}_r(\mathcal{T},\mathcal{U})$ be the Brezzi-Douglas-Marini spaces over $\mathcal{T}$ with normal boundary conditions along $\partial\Omega$. At this point we recall the divergence operator and the vector-valued curl operator; see also the next section for more details.

First we perform the flux reconstruction for the divergence. Let $f_h \in \mathcal{P}_{r,\mathsf{DC}}(\mathcal{T})$ be a function over Ω that is piecewise in $\mathcal{P}_r(\mathcal{T})$ and satisfies $\int_\Omega f_h = 0$. Then there exists $\xi_h \in \mathbf{RT}_r(\mathcal{T},\mathcal{U})$, generally not unique, with $\operatorname{div} \xi_h = f_h$. To compute such a vector field, let $f_h' \in \mathcal{P}_{r,\mathsf{DC}}(\mathcal{T})$ be the L^2 projection of f_h onto the piecewise constant functions. Then $\int_\Omega f_h' = \int_\Omega f_h = 0$, and hence there exists $\xi_h' \in \mathbf{RT}_0(\mathcal{T},\mathcal{U})$ with vanishing normal components along $\partial\Omega$ and $\operatorname{div} \xi_h' = f_h'$. Next we let $f_h'' := f_h - f_h'$. For each $T \in \mathcal{T}^2$ we have $\int_T f_h'' = 0$ by construction; hence there exists $\xi_T'' \in \mathring{\mathbf{RT}}_r(T)$ with $\operatorname{div} \xi_T'' = f''_{h|T}$. We let $\xi_h'' := \sum_{T\in\mathcal{T}^2} \xi_T''$ and $\xi_h := \xi_h' + \xi_h''$. Then $\xi_h \in \mathbf{RT}_r(\mathcal{T},\mathcal{U})$ is the desired flux reconstruction.

Next we show the flux reconstruction for the curl operator. Suppose that $\theta_h \in \mathbf{RT}_r(\mathcal{T},\mathcal{U})$ is the curl of a member of $\mathcal{P}_{r+1}(\mathcal{T},\mathcal{U})$. We let $\theta_h' \in \mathbf{RT}_0(\mathcal{T},\mathcal{U})$ be the canonical interpolation onto the lowest-order Raviart-Thomas space. Since the canonical interpolation commutes with differential operators, there exists $\sigma_h' \in \mathcal{P}_1(\mathcal{T},\mathcal{U})$ with $\operatorname{div} \sigma_h' = \theta_h'$. Let $\theta_h'' := \theta_h - \theta_h'$. Note that θ_h'' has a well-defined normal trace over the edges of $\mathcal{T}$. For every edge $E \in \mathcal{T}^1$ we have $\int_E \vec{n}_E \cdot \theta_h'' = 0$. By taking the antiderivative of $\vec{n}_E \cdot \theta''$ over the edge E and extending the result onto the triangles that contain E, we conclude that there exists $\sigma_E'' \in \mathcal{P}_{r+1}(\mathcal{T},\mathcal{U})$ supported on the two triangles adjacent to E with $\vec{n}_E \cdot (\theta_h'' - \operatorname{curl} \sigma_E'') = 0$. We let $\sigma_h'' = \sum_{E\in\mathcal{T}^1} \sigma_E''$ and let $\theta_h''' := \theta'' - \operatorname{curl} \sigma_h''$. By construction, we can write $\theta_h''' = \sum_{T\in\mathcal{T}^2} \theta_T'''$ where for each $T \in \mathcal{T}^2$ we have $\theta_T''' \in \mathring{\mathbf{RT}}_r(T)$ and $\operatorname{div} \theta_T''' = 0$. For each triangle $T \in \mathcal{T}^2$ there exists $\sigma_T''' \in \mathring{\mathcal{P}}_{r+1}(T)$ with $\operatorname{curl} \sigma_T''' = \theta_T'''$. We set $\sigma_h''' = \sum_{T\in\mathcal{T}^2} \sigma_T'''$. Eventually, we let $\sigma_h := \sigma_h' + \sigma_h'' + \sigma_h'''$ and observe $\operatorname{curl} \sigma_h = \theta_h$.

7. Applications in A Posteriori Error Estimation

In this section we apply the partially localized flux reconstruction to obtain a fully localized flux reconstruction for equilibrated a posteriori estimators. A thorough and comprehensive study of equilibrated a posteriori error estimators from the perspective of exterior calculus will be subject of research in future publications. At this point, we only focus on a simple example and important special case, namely the curl-curl equation over a two-dimensional domain. With our techniques we can solve an open problem: we generalize the equilibrated a posteriori error estimator for Nédélec elements of (14) from the case of lowest-order to case of higher and possibly non-uniform polynomial order.

This section extends Example 6.7 above. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. We let $L^2(\Omega)$ and $\mathbf{L}^2(\Omega) := L^2(\Omega)^2$ denote the Hilbert spaces of square-integrable functions and vector fields, respectively, over Ω. The corresponding scalar products and norms are written $\langle \cdot, \cdot \rangle_{L^2}$ and $\| \cdot \|_{L^2}$, respectively. We let $H^1(\Omega)$ be the first-order Sobolev space and let $\mathbf{H}(\mathrm{div}, \Omega)$ be the space of square-integrable vector fields with divergence in $L^2(\Omega)$. These are Hilbert spaces endowed with the respective graph scalar products of the gradient and the divergence,

$$\mathrm{grad} : H^1(\Omega) \to \mathbf{L}^2(\Omega), \quad \omega \mapsto (\partial_x \omega, \partial_y \omega),$$
$$\mathrm{div} : \mathbf{H}(\mathrm{div}, \Omega) \to L^2(\Omega), \quad (u, v) \mapsto \partial_x u + \partial_y v.$$

Consider the isometry $J : \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega)$ which rotates each vector field by a right angle counterclockwise, i.e. $J(u, v) \to (-v, u)$ for $(u, v) \in \mathbf{L}^2(\Omega)$. We introduce

$$\mathbf{H}(\mathrm{curl}, \Omega) := J^{-1}\mathbf{H}(\mathrm{div})$$

and introduce the differential operators

$$\mathrm{curl} : \mathbf{H}(\mathrm{curl}, \Omega) \to L^2(\Omega), \quad \nu \mapsto \mathrm{div}\, J\nu,$$
$$\mathrm{curl} : H^1 \to \mathbf{L}^2(\Omega), \quad \tau \mapsto J\,\mathrm{grad}\,\tau.$$

We have Hilbert spaces $H^1(\Omega)$ and $\mathbf{H}(\mathrm{curl}, \Omega)$ with the respective graph scalar products. To formalize boundary conditions, we let $H^1_0(\Omega)$, $\mathbf{H}_0(\mathrm{div}, \Omega)$ and $\mathbf{H}_0(\mathrm{curl}, \Omega)$ denote the closure of the compactly supported smooth scalar or vector fields over Ω in $H^1(\Omega)$, $\mathbf{H}(\mathrm{div}, \Omega)$ and $\mathbf{H}(\mathrm{curl}, \Omega)$, respectively. It is easy to see that we have well-defined differential complexes

$$0 \to \mathbb{R} \longrightarrow H^1(\Omega) \xrightarrow{\mathrm{grad}} \mathbf{H}(\mathrm{curl}, \Omega) \xrightarrow{\mathrm{curl}} L^2(\Omega) \to 0, \tag{76}$$

$$0 \leftarrow \mathbb{R} \xleftarrow{\int} L^2(\Omega) \xleftarrow{\mathrm{div}} \mathbf{H}_0(\mathrm{div}, \Omega) \xleftarrow{\mathrm{curl}} H^1_0(\Omega) \leftarrow 0. \tag{77}$$

Here, the differential operators have closed range and both differential complexes are mutually L^2 adjoint (as Hilbert complexes in the sense of (15)). If moreover the domain is simply-connected, then the differential complexes (76) and (77) are exact (see 4). We also note the integration by parts formulas

$$\langle \mathrm{curl}\, \nu, \tau \rangle_{L^2} = \langle \nu, \mathrm{curl}\, \tau \rangle_{L^2}, \quad \nu \in \mathbf{H}(\mathrm{curl}, \Omega), \quad \tau \in H^1_0(\Omega), \tag{78}$$

$$\langle \mathrm{grad}\, v, \nu \rangle_{L^2} = -\langle v, \mathrm{div}\, \nu \rangle_{L^2}, \quad v \in H^1(\Omega), \quad \nu \in \mathbf{H}_0(\mathrm{div}, \Omega). \tag{79}$$

Remark 7.1.
It is a notational inconvenience of two-dimensional vector calculus that two different differential operators are called curl. One curl maps vector fields to scalar functions and the other curl maps scalar functions to vector fields. The *curl* on vector fields is also called *rot* in several publications (4; 3), but there seems to be no universal convention in mathematics.

The curl-curl problem is to find a vector field υ that satisfies $\mathrm{curl}\,\mathrm{curl}\,\upsilon = \theta$ for a given vector field θ. Specifically, we consider a weak formulation over Sobolev spaces of vector fields, where we assume that $\theta \in \mathbf{L}^2(\Omega)$ and search for $\upsilon \in \mathbf{H}(\mathrm{curl}, \Omega)$ with

$$\langle \mathrm{curl}\, \upsilon, \mathrm{curl}\, \nu \rangle_{L^2} = \langle \theta, \nu \rangle_{L^2}, \quad \nu \in \mathbf{H}(\mathrm{curl}, \Omega). \tag{80}$$

Solutions of (80) are generally not unique because the curl operator has non-trivial kernel. To ensure uniqueness one may require the solution υ to be orthogonal to the gradients of functions in $H^1(\Omega)$; one can show that this enforces $\upsilon \in \mathbf{H}_0(\mathrm{div}, \Omega)$ with $\mathrm{div}\, \upsilon = 0$. Conditions to ensure uniqueness of υ, however, are not central to our demonstration in this section.

Assume additionally that $\theta \in \mathbf{H}_0(\mathrm{div}, \Omega)$ with $\mathrm{div}\,\theta = 0$. Then θ is the curl of a scalar function in $H^1_0(\Omega)$. Definitions imply that $\mathrm{curl}\, \upsilon \in H^1_0(\Omega)$ with $\mathrm{curl}\,\mathrm{curl}\, \upsilon = \theta$, and hence every weak solution of (80) is a strong solution in the case of such θ.

In order to address a posteriori error estimation we fix a solution $\upsilon \in \mathbf{H}(\mathrm{curl}, \Omega)$ and let $\upsilon_h \in \mathbf{H}(\mathrm{curl}, \Omega)$ be arbitrary. Furthermore we let $\sigma \in H^1_0(\Omega)$ with $\mathrm{curl}\, \sigma = \theta$. By the binomial theorem we see

$$\begin{aligned} &\|\sigma - \mathrm{curl}\, \upsilon_h\|^2_{L^2} \\ &\quad = \|\sigma - \mathrm{curl}\, \upsilon\|^2_{L^2} + \|\, \mathrm{curl}\, \upsilon - \mathrm{curl}\, \upsilon_h\|^2_{L^2} - 2 \left\langle \sigma - \mathrm{curl}\, \upsilon, \mathrm{curl}\, \upsilon - \mathrm{curl}\, \upsilon_h \right\rangle_{L^2}. \end{aligned}$$

Using (78) and $\mathrm{curl}\, \sigma = \theta = \mathrm{curl}\, \upsilon$ we note

$$\langle \sigma - \mathrm{curl}\, \upsilon, \mathrm{curl}\, \upsilon - \mathrm{curl}\, \upsilon_h \rangle_{L^2} = \langle \mathrm{curl}(\sigma - \mathrm{curl}\, \upsilon), \upsilon - \upsilon_h \rangle_{L^2} = 0.$$

Thus we conclude

$$\|\sigma - \mathrm{curl}\, \upsilon_h\|^2_{L^2} = \|\sigma - \mathrm{curl}\, \upsilon\|^2_{L^2} + \|\, \mathrm{curl}\, \upsilon - \mathrm{curl}\, \upsilon_h\|^2_{L^2}. \tag{81}$$

Equation (81) is a generalized Prager-Synge identity (see 14).

We motivate this result as follows. Let $\upsilon \in \mathbf{H}(\mathrm{curl}, \Omega)$ with $\mathrm{curl}\, \upsilon \in H^1_0(\Omega)$ be a strong solution of (80). Given any exact solution $\sigma \in H^1_0(\Omega)$ of $\mathrm{curl}\, \sigma = \theta$ and any $\upsilon_h \in \mathbf{H}(\mathrm{curl}, \Omega)$, we obtain via (81) that

$$\|\sigma - \mathrm{curl}\, \upsilon_h\|_{L^2} \geq \|\, \mathrm{curl}\, \upsilon - \mathrm{curl}\, \upsilon_h\|_{L^2}. \tag{82}$$

The left-hand side of (82) is given in terms of known objects and dominates the right-hand side of (82), which depends on the generally unknown true solution υ. Seeing υ_h is seen as an approximation of υ, we may see (82) as an error estimate for the derivatives.

In a typical application, υ_h is the Galerkin solution of a finite element method. We can apply (82) to obtain an upper bound on one component of the error in the $\mathbf{H}(\mathrm{curl}, \Omega)$ norm provided that an exact solution $\sigma \in H^1_0(\Omega)$ of $\mathrm{curl}\, \sigma = \theta$ is available. Note that the exact solution $\mathrm{curl}\, \upsilon$ is generally unknown and hence not a candidate for σ. But numerical algorithms for flux reconstruction make (82) productive for applications.

As a technical preparation, we consider finite element de Rham complexes over the domain Ω. Let $\mathcal{T}$ be a simplicial complex triangulating Ω and let $\mathcal{U}$ denote the subcomplex of $\mathcal{T}$ triangulating $\partial\Omega$. The latter is merely a finite set of line segments in this case. We focus on higher order finite element spaces of uniform order; the generalization to spaces of non-uniform polynomial order is straight forward. Let $r \in \mathbb{N}_0$ and recall the Nédélec space $\mathbf{Nd}_r(\mathcal{T})$ of polynomial order r with respect to $\mathcal{T}$. Consider the finite element de Rham complexes

$$0 \to \mathbb{R} \longrightarrow \mathcal{P}_{r+1}(\mathcal{T}) \xrightarrow{\mathrm{grad}} \mathbf{Nd}_r(\mathcal{T}) \xrightarrow{\mathrm{curl}} \mathcal{P}_{r-1,\mathrm{DC}}(\mathcal{T}) \to 0$$

and

$$0 \leftarrow \mathbb{R} \xleftarrow{\int} \mathcal{P}_{r-1,\mathrm{DC}}(\mathcal{T}) \xleftarrow{\mathrm{div}} \mathbf{RT}_r(\mathcal{T}, \mathcal{U}) \xleftarrow{\mathrm{curl}} \mathcal{P}_{r+1}(\mathcal{T}, \mathcal{U}) \leftarrow 0.$$

The first is a finite-dimensional subcomplex of (76) and the second is a finite-dimensional subcomplex of (77).

Let $\theta \in \mathbf{H}_0(\operatorname{div},\Omega)$ be as before but assume additionally that $\theta \in \mathbf{RT}_r(\mathcal{T},\mathcal{U})$. Then there exists a member of $\mathcal{P}_{r+1}(\mathcal{T},\mathcal{U})$ whose curl equals θ. In order to utilize the Prager-Synge identity and estimate (82), it remains to algorithmically construct a generalized inverse for the operator

$$\operatorname{curl} : \mathcal{P}_{r+1}(\mathcal{T},\mathcal{U}) \to \mathbf{RT}_r(\mathcal{T},\mathcal{U}). \tag{83}$$

One possibility is solving a least-squares problem over the whole finite element space. We have seen, however, that a global computation only over lowest-order finite element spaces is sufficient. Using the construction in Example 6.7, we decompose

$$\theta = \theta_0 + \operatorname{curl} \xi_r,$$

where $\theta_0 \in \mathbf{RT}_0(\mathcal{T},\mathcal{U})$ is the canonical interpolation of θ onto the lowest-order Raviart-Thomas space with homogeneous normal boundary conditions and where $\xi_r \in \mathcal{P}_{r+1}(\mathcal{T},\mathcal{U})$ is computed through a number of local problems over simplices whose computation is parallelizable. This reduces the least-squares problem to the special case $r = 0$.

The partially locally flux reconstruction can be extended to a *fully localized* flux reconstruction if additional information about θ is given. Specifically, assume that $\upsilon_h \in \mathbf{Nd}_r(\mathcal{T},\mathcal{U})$ satisfies

$$\langle \operatorname{curl} \upsilon_h, \operatorname{curl} \nu_h \rangle_{L^2} = \langle \theta, \nu_h \rangle_{L^2}, \quad \nu_h \in \mathbf{Nd}_r(\mathcal{T},\mathcal{U}). \tag{84}$$

As above, we compute $\theta_0 \in \mathbf{RT}_0(\mathcal{T},\mathcal{U})$ and $\xi_r \in \mathcal{P}_{r+1}(\mathcal{T},\mathcal{U})$ by local computations such that $\theta = \theta_0 + \operatorname{curl} \xi_r$. Note that $\operatorname{curl} \upsilon_h \in \mathcal{P}_{r-1,\mathrm{DC}}(\mathcal{T})$. Let $\gamma_h \in \mathcal{P}_{0,\mathrm{DC}}(\mathcal{T})$ be the L^2 orthogonal projection of $\xi_r - \operatorname{curl} \upsilon_h$ onto $\mathcal{P}_{0,\mathrm{DC}}(\mathcal{T})$. We note γ_h can be computed for each simplex independently. Thus

$$\langle \gamma_h, \tau_h \rangle_{L^2} = \langle \xi_r - \operatorname{curl} \upsilon_h, \tau_h \rangle, \quad \tau_h \in \mathcal{P}_{0,\mathrm{DC}}(\mathcal{T}).$$

We then find for $\nu_h \in \mathbf{Nd}_0(\mathcal{T},\mathcal{U})$ that

$$\begin{aligned} 0 &= \langle \theta, \nu_h \rangle - \langle \operatorname{curl} \upsilon_h, \operatorname{curl} \nu_h \rangle \\ &= \langle \theta_0, \nu_h \rangle + \langle \xi_r - \operatorname{curl} \upsilon_h, \operatorname{curl} \nu_h \rangle \\ &= \langle \theta_0, \nu_h \rangle - \langle \gamma_h, \operatorname{curl} \nu_h \rangle \end{aligned}$$

because of the Galerkin orthogonality (84) and $\operatorname{curl} \nu_h \in \mathcal{P}_{0,\mathrm{DC}}(\mathcal{T})$. Moreover $\operatorname{div} \theta_0 = 0$ since the canonical interpolator commutes with the differential operators. The next crucial step is to use the the construction of (14). Their results imply the existence of $\varrho_h \in \mathcal{P}_{1,\mathrm{DC}}(\mathcal{T})$ with

$$\langle \varrho_h, \operatorname{curl} \nu \rangle_{L^2} = \langle \theta_0, \nu \rangle_{L^2} + \langle \gamma_h, \operatorname{curl} \nu \rangle_{L^2}, \quad \nu \in \mathbf{H}(\operatorname{curl},\Omega), \tag{85}$$

where ϱ_h can be computed by solving localized problems over element patches around vertices. This leads us to

$$\begin{aligned} \langle \theta, \nu \rangle - \langle \operatorname{curl} \upsilon_h, \operatorname{curl} \nu \rangle &= \langle \theta_0, \nu \rangle - \langle \gamma_h, \operatorname{curl} \nu \rangle + \langle \xi_r - \operatorname{curl} \upsilon_h - \gamma_h, \operatorname{curl} \nu \rangle \\ &= \langle \varrho_h + \xi_r - \operatorname{curl} \upsilon_h - \gamma_h, \operatorname{curl} \nu \rangle \end{aligned}$$

for all $\nu \in \mathbf{H}(\operatorname{curl})$. We set

$$\sigma_h := \varrho_h + \xi_r - \gamma_h.$$

The above results show that

$$\langle \theta, \nu \rangle = \langle \sigma_h, \operatorname{curl} \nu \rangle, \quad \nu \in \mathbf{H}(\operatorname{curl}, \Omega),$$

which implies that $\sigma_h \in H^1_0(\Omega)$. In particular, $\sigma_h \in \mathcal{P}_{r+1}(\mathcal{T}, \mathcal{U})$ with

$$\theta = \operatorname{curl} \sigma_h.$$

The function σ_h has been constructed only by local computations. This completes the flux reconstruction and enables the a posteriori error estimate (82).

Remark 7.2.
Our techniques apply similarly to higher order flux reconstruction for edge elements in dimension three. Again, the lowest-order case is treated of (14).

Remark 7.3.
With Remark 6.4 in mind, we see that ξ_r and γ_h are computable on each simplex using only the information given on that simplex. At the cost of redundant computations, we may rearrange the calculations so that σ_h is constructed with a single parallelizable block of problems associated to patches. Via Remark 6.5 we furthermore see that the stability of the construction of σ_h depends only on the mesh quality, the domain, and the polynomial order of the finite element spaces. We conjecture that the last dependence can be dropped, i.e. that equilibrated a posteriori error estimators for edge elements are robust with respect to the polynomial degree (see 13).

8. Concluding Remarks

In this article we have developed the notion of partially localized flux reconstruction using the framework of finite element exterior calculus. This new tool is of theoretical interest on its own, but our motivating application has been to generalize the equilibrated a posteriori error estimator of Braess and Schöberl to edge elements of higher and possibly non-uniform polynomial order.

There are several directions for future research. Whereas the flux reconstruction has been constructed within the framework of finite element exterior calculus, our application to a posteriori error estimation has considered only edge elements in two and three dimensions. A thorough and comprehensive examination of equilibrated a posteriori error estimators in finite element exterior calculus will be subject of subsequent research. This includes generalizing the lowest-order flux reconstruction of (14) to general Whitney k-forms. Apart from these theoretical considerations, research on the efficient implementation of the Braess-Schöberl estimator on edge elements is of practical interest. The author is not aware of computational studies in this regard.

The partially localized flux reconstruction has built upon investigations on the structure of higher order finite element spaces. We could have carried out the construction only with spaces of uniform polynomial order, but within our abstract framework it has been easy and natural to consider the more general case of finite element spaces of non-uniform polynomial order. Of course, the latter are a vast research topic on their own and of fundamental interest for hp-adaptive finite element methods. Future research will investigate whether finite element exterior calculus can provide new techniques for finding bases of higher order spaces with improved condition numbers (32), sparsity properties (9), or algorithmic features (30; 29).

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Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, NO-0316 Oslo, Norway
Email address: martinwl@math.uio.no